Financial contagion: Robustness of financial network structures

by

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THESIS

for the degree of

MASTER OF SCIENCE

(Master i Anvendt matematikk og mekanikk)

Faculty of Mathematics and Natural Sciences
University of Oslo

May 2011

Abstract

In [2] Allen and Gale showed how an interbank market can be affected by regional liquidity shocks, and how a local bank crisis can become global when banks are interconnected through deposits. The purpose of this thesis is to analyze the robustness of different interbank market structures towards regional liquidity shocks. We formulate a mathematical framework for the analysis and show how to find optimal deposits between banks by solving a minimum flow distribution problem on a network with the underlying graph representing the market structure. We use a breadth-first search (BFS) algorithm, which takes the optimal deposits between banks as input parameter, to traverse through the graph and analyze the effects of a regional liquidity shock.

Central results include showing that a maximum correlation linear graph is the least robust market structure, and we derive an optimality result for the k-regular bipartite graph. Robustness properties of other graph structures are also found. The results are shown by deriving properties of the minimum flow distribution problem, which are used for the analysis of the BFS algorithm.

The thesis also include numerical simulations, which confirm and illustrate the theoretical results, and MATLAB program code written specifically to solve the Allen and Gale model for different underlying graph structures.

Preface

This thesis was submitted as final part of the requirements for the degree *Master of Science* at the University of Oslo (UiO). It has been written at the University of California, Berkeley, where I spent the last year as a visiting student researcher. Having the privilege of writing a thesis under the Californian sun has been a great pleasure and I am grateful to my friend and fellow student, Navreet Kaur, for encouraging me to pursue this opportunity.

The most important contribution to this thesis has been from my advisors, who deserve the greatest acknowledgments. Professor Geir B. Asheim at the Department of Economics at UiO, my main advisor, for helpful advice and comments, and for being patient and responsive despite my being on a different continent throughout the whole process. Professor Shachar Kariv at the Department of Economics at the University of California, Berkeley, for suggesting the particular topic of the thesis and pushing my research in interesting directions. Professor Geir Dahl at the Department of Mathematics at UiO for introducing me to game theory and graph theory, and for help with practicalities as well as input on mathematical aspects of the thesis.

I would also like to thank the study administration at Department of Mathematics at UiO for being generally helpful, making it possible for me to be abroad for study, work and travel when I should have been at the university.

Last but not least I want to thank Øyvind Indrebø for useful discussions and comments on mathematical arguments, Laura Cowell for proofreading and help with English comma rules, Jens Kristian Holmen for great company when working out and studying, Molly Kawahata for awkwardness, and my parents who taught me to read and write, add and subtract; invaluable skills when writing a master's thesis. A special thanks to International House, Berkeley for providing the library, where I have spent endless hours writing, and the patio, where I have spent almost as many hours reading in the sun.

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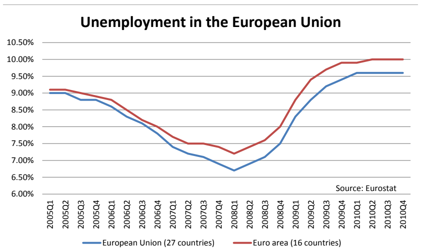
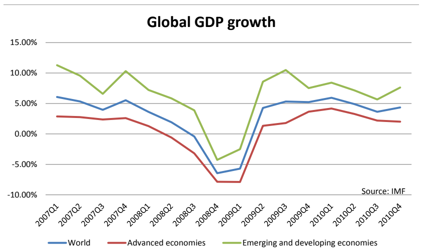
Chapter 1

Introduction

The financial crisis, which struck the world economy in the fall of 2008, was a prime example of how fragile and interconnected financial markets are. What started as a housing crisis in the United States, led to a global breakdown that affected not only Wall Street, but financial institutions, governments and people all over the world.

The scale of damage driven by the crisis was enormous. In 2007 real GDP growth in the industrialized world was 2.8%. In 2009 it had fallen to −3.2%. Public debt levels have increased tremendously and gross government debt in G-20 countries is expected to rise from 78% in 2007 to 118% in 2014 [23]. In the European Union, the number of unemployed increased by 5.4 million from March 2008 to May 2009 [11]. In its latest Global Financial Stability Report, IMF estimates the cost of the crisis in terms of bank losses to be $2.2 trillion [12]. For comparison, this is nearly 5 times the value of the Norwegian Pension Fund Global as of 2010 (est $0.5 trillion) [18].





The World Bank chief, Robert Zoellick, said about the recent financial crisis that "This has been a man-made catastrophe [...] We need concerted action now to build a better system for the future.[1]" In the crisis aftermath there has been great political will to regulate the roles of financial intermediaries and reform the way financial markets operate. Going from political will to implementing actual measures to ensure financial stability must be founded on an understanding of financial markets and financial crises.

Although the severity and far-reaching impact of the recent crisis brought hightened public awareness to the matter, financial crises are well-known features of the financial market. A number of crises have been identified and analyzed throughout history. Sprague surveys crises in the United States during the era of National Banking as early as 1910 [24]. Much has been written on the Great Depression, arguably the largest recession in modern times, e.g. [16] or [5]. After the World War II there was a 25 year period without major crises [4], while the World Bank lists more than 80 crises from 1975 to 1996 [6]. Financial crises have also been analyzed in various theoretical frameworks. In the literature on financial crises, there are two key issues at hand: What happens before and during a crisis, and do such events result from inherent qualities in the financial market. The answers to these questions can potentially have a huge impact on policy making and regulations.

Arguably, the most influential model of banking panics and financial crises is the Diamond-Dybvig model [10]. The model uses banks' investments in illiquid assets and the fact that depositors are allowed to withdraw from a bank at any time to develop a theory of bank runs as an equilibrium phenomenon. The model advocates the view of banks as inherently unstable institutions.

The last decade has seen an enormous growth in the literature on social and economic networks. The growth has been driven partly by increased awareness of the role that networks play in such settings, partly by the enormous amount of data produced from the Internet, and partly from the improved computational possibilities from faster computers and better algorithms. The two latter enable us to reconstruct and analyze much larger real-world networks than ever before. Developing this field of study, the study of complex social and economic networks, has been a joint effort from a multitude of disciplines with important contributions from computer scientists, mathematicians and statisticians, and economists and sociologists. For an overview of the economics of networks, see [15] or [25]. For an introduction to many of the algorithmic and mathematical aspects of network economics, [20] presents up-to-date research topics in the related field of algorithmic game theory.

[1] http://www.telegraph.co.uk/finance/financialcrisis/3187467/

The theory of financial crises has not escaped the attention of network economists. An important contribution to the literature was made by Allen and Gale in 2000 [2]. Their "Financial Contagion" is one of the first papers to discuss the role the network structure can play in transmitting a crisis. Building on the Diamond-Dybvig model, they analyze an interbank market connected by deposits held by banks in other banks. The paper's most important conclusions are that the connectedness of the market can lead to the spread of a financial crisis to other regions through spillover effects between the regions, and that the structure of interconnections critically affects the potential spread of a crisis.

The theoretical literature on contagion takes two approaches. Following the paper from Allen and Gale, many models have explained various network effects created from individual bank risk. The source to contagion in all this work is the direct linkages between banks, like the interbank deposits in [2]. A different approach is one where the focus is indirect linkages in the form of dependencies between banks' portfolios. For an extensive survey of literature on financial contagion, see [1].

The literature on contagion seems to agree that dense networks with a high number of connections and short diameters are more robust than less dense networks. When it comes to the effect of different network topologies beyond this high connectivity property, the literature is very limited. This thesis addresses this void, and tries to provide some understanding on how the network structure affects the robustness of financial markets. We use Allen and Gale's model as a starting point to explore a broader range of network structures and their properties in equilibrium.

In Chapter 2, the reader is introduced to the basics of game theory, graph theory and financial intermediation. The purpose is to provide a theoretical context for the thesis and to introduce concepts that will be used in later chapters.

Chapter 3 is an extensive presentation of Allen and Gale's model of financial contagion and a summary of the most important results they present in [2]. The economic intuition behind the model as well as more technical aspects are explained in this chapter.

The analyses in the original article are restricted to two symmetric graphs. In Chapter 4, we introduce a mathematical generalization of the problem that will enable us to analyze a broader range of graph families. In particular we see how the problem of redistributing liquidity in an interbank market can be formulated as a minimum network flow problem and how the spread of a crisis can be analyzed with a breadth first search algorithm.

Chapter 5 contains the main theoretical results developed in this thesis. It is shown that the linear graph with maximum correlation between nodes is the least robust graph. An optimality property of a k-regular bipartite graph in terms of robustness is derived. Robustness properties of certain other graph structures are also presented. All analyses are undertaken within the framework from Chapter 4.

In Chapter 6, we provide numerical examples and simulations both to confirm our theoretical results and illustrate the dynamics of the model.

Finally we conclude in Chapter 7 by summing up the insights from this thesis and pointing out possible directions for further research within the field.

Chapter 2

Theoretical background

This chapter will introduce basic theory in topics relevant for the thesis such as game theory, graph theory and financial intermediation. The purpose is to provide a context within which the thesis is studied and develop notation and results that are used in later chapters.

2.1 Game theory

Game theory can be described as the study of strategic interaction. More precisely, it is a branch of mathematics studying situations with multiple players, where all players choose between a set of strategies, and each player's payoff to choosing a strategy depends on the strategies chosen by the other players. Definitions and results are from [26], which gives a much more extensive introduction to basic game theory. Computational aspects of game theory, which is of great general interest, but of limited relevance for this thesis, are dealt with in the article collection [20].

There are two equivalent standard representations of games. We first define the *normal form game*.

Definition 2.1. A normal form game Γ consists of a set of n players, $\{1, 2, \ldots, n\}$. Each player i has a set of possible strategies, S_i. The set of all possible combinations of strategies is S where S is the Cartesian product $S_1 \times S_2 \times \cdots \times S_n$. Player i chooses a strategy $s_i \in S_i$. The vector $s = (s_1, \ldots, s_n)$ of all the players' strategies is called a *strategy profile*. Player i's preferences over outcomes is represented by a utility function $u_i : S \to \mathbb{R}$.

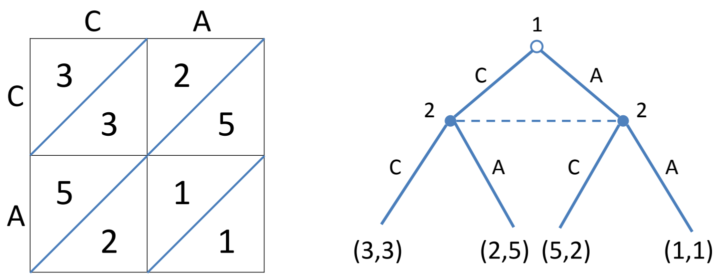
The vector of strategies played by all players except player i is for shorthand written $s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$.

A game can also be represented as an *extensive form game*.

Definition 2.2. An extensive form game Γ consists of the following data:

1. A finite set $\{1, 2, \ldots, n\}$ of players.
2. A rooted tree T called a game tree with a payoff for each of the n players associated to each leaf node.
3. A partition of non-leaf nodes in $n+1$ subsets, one for each player, and a subset for "Nature" with a probability distribution over outgoing edges.
4. Nodes of Nature have a probability distribution over outgoing edges.
5. A partitioning of a player's set of nodes in information sets (defined below).

For complete information games, the subset of Nature is empty. The description of the game is common knowledge among the players.



(a) Normal form game (b) Extensive form game

Figure 2.1: Prisoners dilemma - different representations

A strategy for a player in an extensive form game, is a complete specification of which action to take at every node belonging to the player.

In extensive form games, the notion of *information sets* plays an important role.

Definition 2.3. An information set in an extensive form game Γ is a collection of decision nodes such that

1. The same player is assigned the move at every node in the set.
2. The player has the same available action choices at every node.
3. The moves played to reach a node in the information set is the same for all nodes in the set.

Central in the study of game theory is equilibrium analysis. Informally, an equilibrium is a situation where all players have adopted strategies that they are unlikely to change. There are many different equilibrium concepts which try to capture this idea. Important for this thesis and arguably the most well-known is the *Nash Equilibrium*. A strategy profile s is a Nash Equilibrium if the following holds for all players: A player cannot increase his own payoff by changing strategy, given that the other players stick to their strategies. More formally, we define pure strategy and mixed strategy equilibriua.

Definition 2.4. A strategy profile $s^* \in S$ is a *Nash equilibrium* in pure strategies if for all players i and all strategies $s_i \in S_i$ the following holds:

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*).$$

To define the notion of a *mixed strategy Nash Equilibrium* we need to first define a *mixed strategy*

Definition 2.5. A mixed strategy x_i for player i is a probability distribution over the set of player i's possible strategies S_i. Let X_i be the set of mixed strategies for player i. Define by x a mixed strategy profile $x = (x_1, x_2, \ldots, x_n)$ and the set of all mixed strategy profiles $X = X_1 \times X_2 \times \ldots \times X_n$. Furthermore, define

$$x(s) = \Pi_{j=1}^{n} x_j(s_j),$$

as the probability of combination s under the mixed strategy profile x. The expected payoff to player i under a mixed strategy profile x is

$$U_i(x) = \sum_{s \in S} x(s) u_i(s).$$

Definition 2.6. For a game Γ, a strategy profile $x = (x_1, x_2, \ldots, x_n)$ in X is a mixed strategy Nash Equilibrium if for every player i and for every mixed strategy $y_i \in X$ we have

$$U_i(x_{-i}, x_i) \geq U_i(x_{-i}, y_i).$$

One of the most important game theoretical results is the existence of a Nash equilibrium in any game with a finite set of players and finite set of strategies.

Theorem 2.1. *If* Γ *is a game with a finite set of players and a finite set of strategies, there exists a Nash equilibrium in mixed strategies.*

Proof. See [19] for Nash's original proof. □

A game can have multiple Nash equilibria. A simple illustration is a co-ordination game with two players. The players each possess one unit of a good and can choose to invest it (I) or not invest it (NI). The investment yields positive net returns if both players invest. If only one player chooses to invest, the investment does not pay off. In this case there are two pure strategy equilibria, a) both players invest, and b) no player invests. We see in

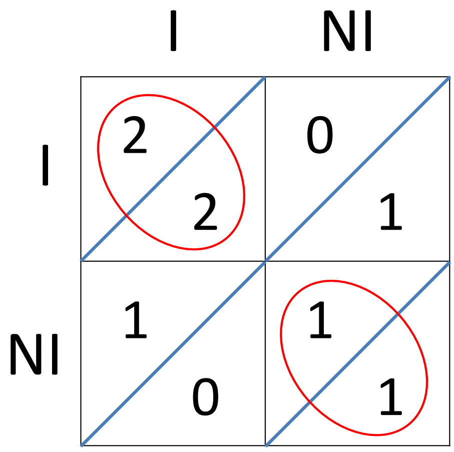


Figure 2.2: Example of coordination game (NE with red circle)

figure 2.2 that the equilibrium where both players invest would yield higher payoffs for both players than the equilibrium where no player invests. When an equilibrium gives higher payoff than another equilibrium for all players, we say that the higher payoff equilibrium *pareto dominates* the other.

Definition 2.7. If s^* and s' are two strategy profiles that are Nash equilibria in a game Γ, we say that s^* pareto dominates s' if, for all players i, the following is satisfied

$$u_i(s^*) \geq u_i(s'),$$

and we have strict inequality for at least one i.

The definition of Nash equilibrium is broad in the sense that games often have multiple Nash equilibria and we can, in certain cases, have equilibria that require behavior that seems irrational or unrealistic from rational utility-maximizing agents. There are many refinements of the equilibrium concept reducing the set of equilibria. An important and widely used refinement, which will be used in this thesis, is the *subgame perfect equilibrium*.

Definition 2.8. Let Γ be a game on extensive form. A subgame is any part of the game that satisfies the following criteria:

1. The initial node is in a singleton information set.
2. It contains every successor node to this node.
3. If it contains any part of an information set, then it contains all nodes in the information set.

Definition 2.9. A strategy profile s is a subgame perfect equilibrium if, for every subgame of the original game Γ, it is a Nash Equilibrium.

The set of subgame perfect equilibria is a subset of Nash Equilibria.

Subgame perfect equilibrium filters out Nash equilibria that could arise from non-credible threats, since it requires the strategies to be consistent in all parts of the game. This could be understood by the ultimatum game, a sequential game where two players divide a sum of money. Figure 2.3 is an example of an ultimatum game on extensive form. The first player proposes how to divide (e.g. either fair (F) or unfair (UF)), and the second player can either accept (A) or reject (R) the offer. If he rejects, both players get nothing, if he accepts, the money is divided according to the first player's proposal. There are two Nash equilibria in this game: One where the second player rejects all offers if the first player get anything at all, and then both players get nothing. The second one is where the second player accepts an offer above a certain threshold and the first player offers the threshold value to the second player. The first Nash equilibrium does not satisfy the stronger subgame perfect equilibrium, since the threat from the second player of rejecting any offer is irrational for a utility maximizing agent, who will always choose the strategy that gives the highest payoff. The threat of rejecting an offer which would yield a positive net payoff is therefore not credible. The second Nash equilibrium is the game's only subgame perfect equilibrium. In figure 2.3 the first Nash equilibrium corresponds to the lower right node and the second subgame perfect equilibrium is the lower left node of the decision tree.

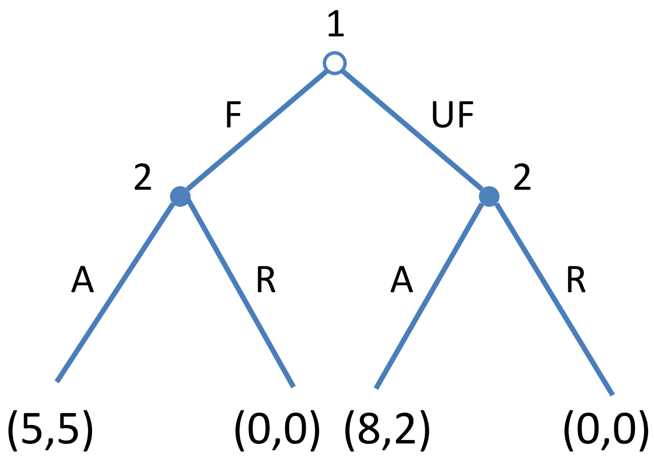


Figure 2.3: Example of ultimatum game on extensive form

2.2 Economics of information

The overview in this section is based on [17], which provides a detailed presentation of the theory of incentives under asymmetric information.

We consider a situation where there are two types of players; principals and agents. A principal is a person who authorizes another person, an agent, to act on his behalf. The principal will seek to maximize the utility from hiring an agent, that is the optimal solution to a principal-agent problem is utility maximizing for the principal. In many strategic situations information has economic value and economic agents might not have access to the same information. This information asymmetry needs to be taken into account when such situations are analyzed and this may affect the optimal solution as opposed to an optimum under full information.

A situation where information asymmetry might cause problems is the principal agent problem. When a principal hires an agent to pursue his interests, it is possible that the agent has conflicting interests. This might result in behavior that is suboptimal, at least from the principal's point of view. A problem that often arises is that of *adverse selection*, which occurs when agents can be of different types, but the principal cannot control for an agent's type. Asymmetric information might lead to a "bad" selection of agents accepting the contract.

In this thesis we will consider a variant of the principal-agent problem, where banks (principals) offer a contract to consumers (agents) that they either accept or reject. Accepting means that consumers place deposits in a bank, such that the bank can invest the deposits in short term and long term assets. The consumers can choose to liquidate their deposit either earlier or later, depending on their preferences. A central planner seeks to find the contract that maximizes consumer utility under certain *feasibility* constraints. The feasibility constraints are simply inherent limitations of the problem. For example the investment problem, where a feasibility constraint is that you cannot invest more than you have. The *first-best allocation* is the solution to the utility maximization problem under the feasibility constraints.

Under information asymmetry, implementation of the first-best allocation is not always possible. When there are different types of economic agents and an agent's type cannot be verified, we can have situations where an agent does not have proper incentives to reveal his own type, but will benefit from mimicking a different type of agent. Conditions to ensure that economic agents reveal their true type are called *incentive compatibility constraints*. An *incentive-efficient* allocation is the optimal allocation when the incentive compatibility constraints are added to the problem. The first-best allocation is not necessarily incentive-efficient, since we, by introducing new constraints to the problem, optimize over a subset of the original domain. If the incentive compatibility constraints are binding, the optimal solution will shift and the information asymmetry will have imposed a cost compared to the first-best that could have been achieved under full information. In this thesis, as we will see in a later chapter, the first-best allocation is also incentive-efficient, so the first-best can be implemented.

2.3 Financial markets

The main interest in this thesis is to analyze a model for financial contagion in an interbank market with respect to the network structure of the market. This section explains aspects of financial markets of interest for the thesis, in particular the role of financial intermediaries like banks, and the theory behind occurrence of financial crises. For an extensive survey of the literature on financial intermediaries and crises, see [14].

2.3.1 Financial intermediation

Financial intermediaries are institutions that borrow from one group (consumers/savers) and lend to another group (companies/investors) that needs resources for investment. In other words, they connect surplus agents with deficit agents and enable the channeling of funds between them.

An important feature of financial intermediaries is that they facilitate the use of consumer liquidity surplus to finance long-term investments. When consumers choose to lend surplus funds to a financial intermediary, the intermediary gets access to liquidity that it can lend to those who need liquidity for some sort of investment. There is a long term perspective to a large share of the investments, to which financial intermediaries provide liquidity. Companies typically borrow to invest in equipment, buildings or infrastructure, which do not yield returns in the short run. Individuals borrow to invest in real property. In none of the cases can the financial intermediary expect to liquidate the loan prematurely, which is normally reflected in the contract between the financial intermediary and the borrower. This use of short-term consumer surplus of liquidity to finance long-term investments has a very important implication; if too many lenders simultaneously choose to liquidate their claims to the financial intermediary, it will be impossible for the intermediary to meet the demand. The system will, in other words, work as long as the financial intermediary is able to predict consumers' behavior within some margin of error when choosing where to channel its liquidity.

Following the discussion above, we see the importance of consumer trust in the financial intermediary. As long as lenders believe that banks are able to meet demand for liquidity at any time, we can expect the lenders to act predictable and rational on average. On the other hand, if depositors for some reason believe that the financial intermediary might be running out of liquid means, and that their holdings in the intermediary are at risk, they are more likely to withdraw. If enough depositors withdraw, the intermediary will run short on liquidity, which again can result in even more depositors withdrawing. This effect is known as a bank run. In a bank run, the high number of depositors withdrawing, because they believe a bank is insolvent, creates a self-fulfilling prophecy; their withdrawal increases the likelihood of a default, which in turn encourages more depositors to withdraw. The result for the bank can be bankruptcy. In this thesis, the starting point for the analysis is a similar situation where a liquidity shock results in bankruptcy. An explanation for the liquidity shock could be a loss of trust in banks. The underlying reasons for a liquidity shock is, however, beyond the scope of this thesis, in which we will consider the network effects given the initial shock.

There are several explanations as to why financial intermediaries exist. Diamond [9] explained financial intermediaries as monitors of borrowers. A lender must monitor a borrower, since there is an ex post information asymmetry in the output of a borrower's investments: Only the borrower knows the real output. From a lender's viewpoint, monitoring borrowers is costly, and it is efficient to delegate the task to a specialized agent, the financial intermediary such as a bank.

Another explanation is that bank-like financial intermediaries can produce costly but valuable information. The basis for the theory is the existence of costly information about investment opportunities, such that economic agents may want to produce this information. This contains, however, a "reliability problem" to it, as it may be difficult for the economic agent to prove the value of the information he has produced. A related problem is that buyers of information can share the information with others without necessarily reducing the value the information has to them. It may be difficult for the information producer to fully capture the returns to producing the information. This is called the "appropriability problem". Both these problems can motivate the existence of an intermediary.

The explanation that is most relevant for this thesis is that banks act as consumption smoothers, an idea first formalized in the Diamond and Dybvig model [10]. Consumers invest liquidity surplus in banks. The model assumes that consumers have uncertain preferences for future consumption. This uncertainty may lead to consumers having to liquidate investments prematurely in order to finance early consumption. If consumers save via intermediation they might be able to diversify the consumption shocks. This also provides a view of banking from the liability side, where consumers have the right to withdraw from the bank to satisfy consumption needs. A third element incorporated in the model is that consumers have private information about the realization of their consumption preferences. An important feature of the Diamond and Dybvig model is the fact that it has multiple equilibrium solutions, one of which is a bank run.

2.3.2 Financial crisis

When we talk about financial crises, bank runs, contagion and related terms, we need to have a clear understanding of what we are talking about. It turns out to be difficult to give precise definitions; the broad literature on the topic does not provide uniform answers. Rather than trying to give generally valid definitions, we will define the terms in the context of this thesis and the model we are working with.

A *bank run* is when a large number of depositors all of a sudden choose to liquidate their deposits in a bank because they believe the bank is or might become insolvent. This could happen in one bank in one region or in many banks in many regions. In this thesis, since banks in a region are assumed to be identical, bank runs will be regional. A bank run may lead to insolvency and bankruptcy.

A *financial crisis* is usually characterized by one of the three following events:

1. High proportion of banks fails.
2. Large/important banks fail.
3. Government intervention needed to prevent 1 or 2.

In this thesis, where we assume numerous identical banks in a region, a regional financial crisis is when a bank run causes bankruptcies in banks in a region.

Contagion is the transmission of a financial shock from one region to other regions. Contagion can occur if banks in a region go bankrupt because of a liquidity shock, and banks in other regions become insolvent and go bankrupt as a result because of financial interdependencies between regions. In the model, when the crisis spread, the spillover effect grows bigger for every new bankrupt region, and will, if it starts spreading, in most cases reach all regions of the world: A regional crisis thus causes a global crisis where all regions are bankrupt.

A central question is whether financial intermediaries are inherently unstable. There are a lot of different theories to support either view. The Diamond and Dybvig model explains the occurrence of financial crises by "sunspot" phenomena, that is external and unpredictable events that make depositors withdraw from the bank. This creates distrust in the bank's ability to provide liquidity among depositors, and the depositors beliefs become self-fulfilling, with a bank run and potentially a financial crisis as the outcome. The model has multiple equilibria, and a sunspot event can move the system from an equilibrium where banks are solvent to one of a bank run [10].

Allen and Gale build on the Diamond and Dybvig model and extend it to a network setting. Banks are interconnected through a network of deposits, and Allen and Gale show that sunspot events in one region can affect other regions, even if the beliefs in those regions stay unchanged. The next chapter gives an extensive presentation of Allen and Gale's model and results.

2.4 Graph Theory

The thesis discusses how the market structure in an interbank market of deposits affects the robustness of the market when exposed to liquidity shocks in a subset of regions. The market is modeled as a graph, and this section will introduce the reader to basic graph theoretic notions that will be used throughout the thesis. This includes the definition, representation, and basic properties of graphs. We will also introduce certain graph families that will be discussed in this thesis.

2.4.1 Basic graph theory

Notation and definitions in this section follow J.A. Bondy and U.S.R. Murty in [3].

A *graph* is a graphical representation of a set of elements called *vertices*, and their interconnections, called *edges*. Edges can be used to illustrate the relation between elements. Vertices are also referred to as *nodes*. *Links* and *connections* are sometimes used to describe edges.

Formally, a graph G is an ordered pair of two disjoint sets, $(V(G), E(G))$. A set $V(G)$ of vertices and a set $E(G)$ of edges. When there is no ambiguity we normally write $V := V(G)$ and $E := E(G)$. Each edge e in G is associated with an unordered pair of vertices u and v in G by an *incidence function* ϕ_G, such that $\phi_G(e) = \{u, v\}$. The edge e is then said to be *incident* with u and v, and we write $e = uv$. If an edge is incident with only one vertex, it is called a *loop*. In this thesis, we consider only *simple graphs*, meaning graphs with no loops. Two vertices incident with the same edge are said to be *adjacent*. The number of vertices in a graph G is denoted $n = |V(G)|$, and the number of edges is $m = |E(G)|$. We are often interested in the set of edges that connects nodes in a subset $U \subset V$ of the nodes with nodes outside the subset. This set is called the boundary of U and is denoted $\partial(U)$. The number of edges with one end in the subset is $|\partial(U)|$.

Graphs have a visual representation, but for many purposes it is useful to introduce matrix notation. We consider two different matrices associated with a graph; the incidence matrix and the adjacency matrix. If $G = (V, E)$ is a graph, the *incidence matrix* of G is the $n \times m$ matrix $M_G := (m_{ve})$ where m_{ve} is the number of times that vertex v and edge e are incident. If the graph has no loops, all entries in the incidence matrix are either 0 or 1. The *adjacency matrix* of G is the $n \times n$ matrix $A_G := (a_{uv})$, where $a_{uv} = 1$

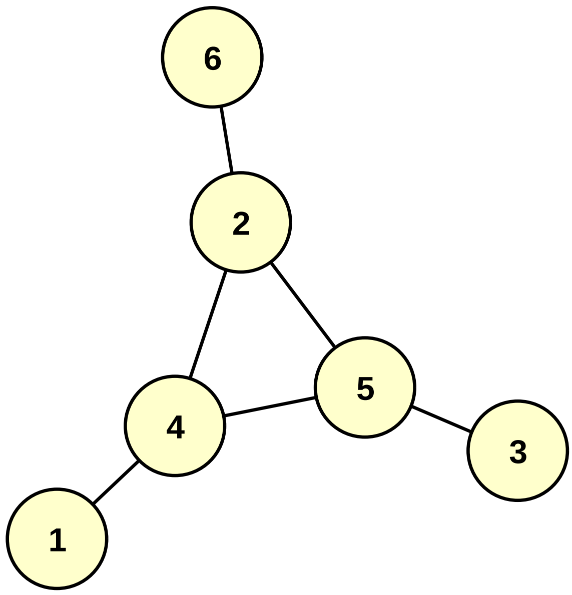


Figure 2.4: Graph with six vertices and six edges

if there is an edge $e \in E(G)$ that connects u and v, and $a_{uv} = 0$ if there is no such edge. Most graphs have more edges than vertices, so the adjacency matrix normally gives the most compact representation.

A graph is *undirected* if the edges have no orientation, that is $e = uv = vu$. The set of adjacent vertices to a vertex v is called the neighbors of v, and is denoted $N_G(v)$. The number of neighbors to a vertex u is the degree of the vertex, denoted $d_G(u)$. $\max_{v \in V} d_G(v)$ is called the *maximal degree*, and $\left(\sum_{v \in V} d_G(v)\right)/n$ is the *average degree* of G.

Certain families of graphs are of particular importance. A graph is *complete* if any two vertices are adjacent. If the edge set is empty, the graph is *empty*. A *path* is an alternating sequence of incident vertices and edges where all edges are distinct. A *cycle* is a path with the the same start and end vertex. If the vertex set can be divided in two disjoint subsets X and Y such that every edge has one end in X and one end in Y, the graph is *bipartite*. If the nodeset consists of two disjoint sets X, Y, such that $X \cup Y = V$, and we can find an edge subset $E' \subset E$ such that each node in X is uniquely linked to a node in Y, we say that the graph $M = (X, Y, E')$ is a *perfect bipartite matching*.

A *subgraph* of $G = (V, E)$ is a graph with node set $V' \subseteq V$ and where the edge set is the restriction of E to V'. A *connected component* in a graph G

is a subgraph of G such that there is a path between any two nodes in the subgraph, and the subgraph is connected to no other nodes in G. If G has only one connected component, G is *connected*. A node v in G is said to be a *cut node* if removal of v and all its incident edges from G gives a graph with more connected components than the original graph.

An important graph family is *trees*. A tree can be defined by the following property

Definition 2.10. A connected graph $T = (V, E)$ is a tree if and only if $|E| = |V| - 1$.

Each node in a tree has an associated set of nodes called *children* and an associated single node called *parent*. A *root* in a tree is a node with no parent node. Nodes with no children are called *leaf nodes*. A subtree of T consists of a node and all its descendants in the tree.

2.4.2 Directed graphs and network flow

In many applications it is useful to allow the connections between nodes to have an orientation. A *directed graph* D, abbreviated *digraph*, is an ordered pair $(V(D), A(D))$ where V is the set of nodes and A a set of *arcs* and an incidence function ϕ_D mapping each arc to an ordered pair of nodes in in V. If a is an arc and $\phi(a) = (u, v)$ we say that u dominates v. The node u is the *tail* of a and the node v is the *head*. We define the negative of a as the arc between the same nodes but opposite orientation; if $a = uv$ then $-a = vu$. If the orientation of an arc is irrelevant for a discussion, it can be referred to as an edge in the digraph. Many definitions and results from the theory of undirected graphs are easily extended to directed graphs. The set of nodes that dominate v is the set of *in-neighbors* of v, and is denoted $N^-_D(v)$. The set of nodes dominated by v is the set of *out-neighbors* of v and is denoted $N^+_D(v)$. Similarly, the cardinality of $N^-_D(v)$ is the *indegree* of v denoted $d^-_D(v)$, and the cardinality of $N^+_D(v)$ gives the *outdegree* of v, $d^+_D(v)$. For a digraph D we can define the *underlying graph* G by replacing each arc $a = uv$ in A by an edge $e = uv$. We write $G(D)$ for the underlying graph of D. Conversely, for a given graph $G = (V, E)$, we define the *associated digraph* by replacing each edge $e = uv \in E$ with two arcs $a = uv$ and $-a = vu$. The digraph associated with G is denoted $D(G)$.

A digraph D is said to be *connected* if the underlying graph $G(D)$ is connected. If there is a path from any node to any other node in D, the digraph is *strongly connected*.

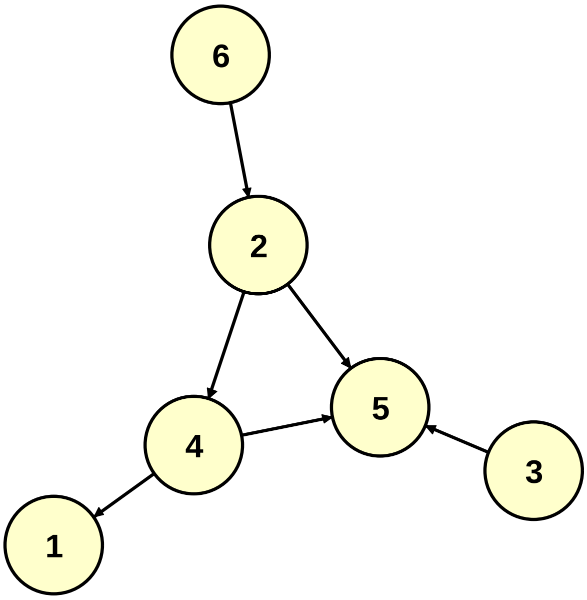


Figure 2.5: Example of digraph with six vertices and six arcs

For analyses in this thesis it is useful to define a *network*.

Definition 2.11. A network $N = N(X, Y)$ is a digraph $D = (V, A)$ with two disjoint node sets, a set $X \subset V$ of *sources* and a set $Y \subset V$ of *sinks* as well as a non-negative *capacity function* $c : A \to \mathbb{R}$. Sources $v \in X$ can be seen as supply nodes and sinks $v \in Y$ as demand nodes. Nodes that are neither sources nor sinks are called *intermediary nodes* and the set of these nodes are denoted I where $I = V \backslash (X \cup Y)$.

Let $f : A \to \mathbb{R}$ be a real-valued function defined on the arcset of a digraph $D(V, A)$. We introduce the following notation: For $S \subset A$, let $f(S) = \sum_{a \in S} f(a)$. We let $\partial^+(X)$ denote the set of arcs with tail in X and head in $V \backslash X$, and $\partial^-(X)$ the set of arcs with head in X and tail in $V \backslash X$. We then define $f^+(X) = f(\partial^+(X))$ and $f^-(X) = f(\partial^-(X))$.

We say that f is a *flow* in $N(X, Y)$ if

$$f^+(v) = f^-(v) \text{ for all } v \in I.$$

That is for any node that is not a sink or a source, the incoming flow equals the outgoing flow. Since each arc has a capacity $c(a)$ we require flow to satisfy the capacity constraint

$$0 \leq f(a) \leq c(a) \text{ for all } a \in A. \tag{2.1}$$

In certain problems, like in this thesis, we require sources to "send" and sinks to "receive" a certain flow. We define $k : V \to \mathbb{R}$ as the function of required net flow from sinks and sources. We can then add two feasibility constraints

$$f^+(v) - f^-(v) = k(v) > 0 \;\forall\, v \in X, \tag{2.2}$$
$$f^+(v) - f^-(v) = k(v) < 0 \;\forall\, v \in Y. \tag{2.3}$$

A flow is *feasible* if the two constraints 2.2 and 2.3 as well as the capacity constraint 2.1 are satisfied.

We also introduce a related concept, *circulation*. Define the real-valued function $f : A \to \mathbb{R}$ on a digraph D. We say that f is a circulation if for every vertex $v \in V(D)$

$$f^+(v) - f^-(v) = 0,$$

that is inflow equals outflow for all vertices $v \in V(D)$.

2.4.3 Network characteristics

To characterize large, complex networks it is useful to define concepts that describe features of the network topology that seem to arise frequently. Such descriptions might enable us to categorize and order networks accordingly. The definitions in this section are taken from [25].

Distance

Distance measures are often important in graphs. The *geodesic distance* $d(i, j)$ is defined as the minimum number of links separating node i from node j. In a graph both the average distance and the maximum distance are of interest. To define those two measures, it is useful to introduce a probability distribution specifying the fraction $\bar{\omega}(r)$ of node pairs at distance r

$$\bar{\omega}(r) = \frac{|\{(i, j) \in N \times N : d(i, j) = r\}|}{n(n-1)}. \tag{2.4}$$

We can then define the *average network distance* to be

$$\bar{d} = \sum_{0<r<\infty} r\bar{\omega}(r), \tag{2.5}$$

and the *diameter* of the network to be

$$\hat{d} = \max\{r : \bar{\omega}(r) > 0\}. \tag{2.6}$$

Connectivity

Another important property of a graph is the degree to which it is interconnected. We talk about two types of connectivity, *node connectivity* and *edge connectivity*.

Definition 2.12. Let $G = (V, E)$ be a connected undirected graph. The subset $V' \subset V$ is a vertex cut if and only if removal of the nodes in V' renders the graph G disconnected. The vertex connectivity $k(G)$ is the size of the smallest vertex cut.

The definition of edge connectivity is analogous.

Definition 2.13. Let $G = (V, E)$ be a connected undirected graph. The subset $E' \subset E$ is an edge cut if and only if removal of the edges in E' renders the graph disconnected. The edge connectivity $\lambda(G)$ is the size of the smallest edge cut.

Connectivity is an important concept in robustness analyses of graphs. An example is found in [7], where the authors argue that the most robust communication network for military applications is one with *optimal connectivity*, which is when $k(G) = \lambda(G) = d_{min}$ where d_{min} is the minimum degree of G.

Clustering

An important feature of a network is its *clustering*. For each node, i, its clustering coefficient C^i is defined as the fraction of pairs of neighbors that are themselves neighbors. Formally, for the network $G = (V, E)$ and any node $i \in V$, the clustering is given by

$$C^i \equiv \frac{|\{jk \in E : ij \in E \text{ and } ik \in E\}|}{\frac{d(i)(d(i)-1)}{2}}. \tag{2.7}$$

In terms of the adjacency matrix A, it can be written as

$$C^i = \frac{\sum_{j<k} a_{ij} a_{ik} a_{jk}}{\sum_{j<k} a_{ij} a_{ik}}. \tag{2.8}$$

If node i has less than two neighbors, we define $C^i \equiv 0$. We can then define the clustering coefficient of the network as the average clustering over all nodes in the network

$$C = \frac{1}{n} \sum_{i=1}^{n} C^i. \tag{2.9}$$

Cohesiveness

Cohesiveness applies to arbitrary subset of nodes in a network. A set is cohesive if the proportion of links that go out of the set is low, i.e. the nodes in the set are for the most connected to each other. Let $U \subset V$ be a subset of nodes in V. For each node i in U we calculate the fraction of links that lie within U

$$H^i(U) = \frac{|\{ij \in E : j \in U|}{d(i)}. \tag{2.10}$$

The overall cohesiveness of the set is then defined as the minimum fraction across all nodes in U

$$H(U) = \min_{i \in U} H^i(U). \tag{2.11}$$

2.4.4 Graph rewiring and small world networks

When networks are large and irregular, so-called complex networks, it is useful to define certain high level characteristics to describe the network structure.

Often when we are interested in analyzing the role network structure plays, like in this thesis, the effect of adding or removing edges from a graph is more or less trivial. The effect of changing the structure of connections by having a fixed number of edges and vertices by "rewiring" the edges, that is moving them to other vertices, is usually complicated to analyze, but could be of great interest.

The process of graph rewiring can be done in various ways, but a standard approach that will be used for simulations in this thesis, is to start with some sort of regular graph, and then to move the end of each edge $e = (u, v)$ in the graph with probability p from v to a vertex w chosen uniformly at random.

Arguably the most used rewiring model is the Watts and Strogatz model, a rewiring procedure used to produce graphs with small-world properties, including short average path length and high degree of clustering [27]. Given a graph of size n and average degree k, the algorithm is as follows:

1. Generate a k-regular ring lattice.
2. For every edge $e = (u, v)$, rewire it with probability p to $e = (u, w)$ where w is chosen with uniform probability from all vertices that avoid loops and link duplication.

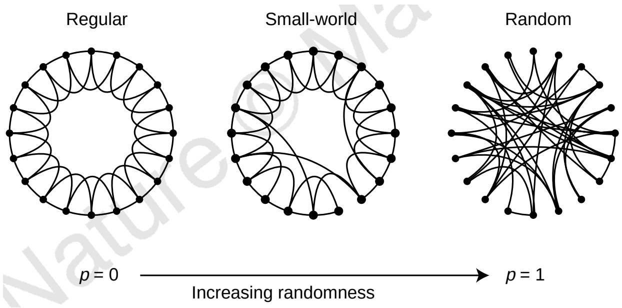


Figure 2.6: Lattice rewired with different probabilities

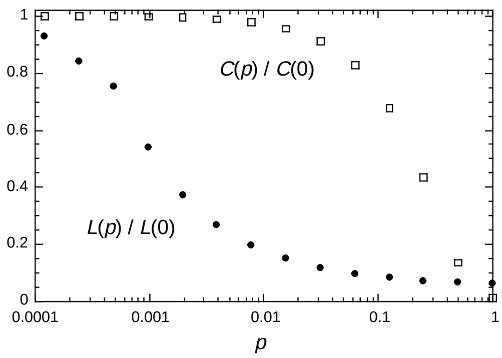


Figure 2.7: Clustering and diameter as function of rewiring probability

A rewiring probability of 0 returns the k-regular ring lattice, while $p = 1$ gives a random graph. For the right choice of p we get graphs with a high degree of clustering, but a low average distance. These graphs play an important role in modeling real-world networks, since networks with such properties are often observed empirically. Graphs with these properties are called small world networks. Figures 2.6 and 2.7 are taken from Watts and Strogatz original paper [27].

Chapter 3

The model

The thesis builds on a model of financial contagion in networks introduced by Allen and Gale in [2]. In this chapter we will present the model and the most important theoretical results. Some aspects relevant for the thesis will be presented and discussed in more general terms than what is done in the original paper. The notation will deviate slightly from Allen and Gale to ensure consistency with notation used for analyses in later chapters.

The model explains how a regional liquidity shock, can result in bankruptcy not only in that region, but, under certain conditions, spread to other regions as well. This is possible because regions are interconnected through claims that banks in different regions have on each other.

We consider a world of banks and consumers, where consumers hold deposits in banks. The banks invest the deposits in a portfolio of short- and long-term assets, such that they can meet the expected demand for liquidity at different time steps by liquidating the short and long-term asset respectively. Aggregate consumer demand for liquidity is known, but there is regional uncertainty with regards to consumer liquidity preferences. The problem of meeting the demand for liquidity at different time steps is therefore a problem of redistributing liquidity from regions with low demand to regions with high demand. By holding deposits in other regions, banks can ensure that the necessary redistribution can be achieved. When a bank faces high demand for liquidity, it can liquidate bank deposits in other regions to meet demand.

As long as the actual demand for liquidity equals the expected demand for liquidity, the interbank market of deposits ensures the necessary redistribution of liquidity, and the demand can be met. But what happens if an unexpected event, a liquidity shock, occurs, and the demand for liquidity exceeds the expected demand.

Allen and Gale's model provides a formal framework to analyze the effects of a liquidity shock in a region on the broader economy. In their paper, they analyze the model in a four-region world for two symmetric market structures. It is shown that even a small perturbation in terms of a liquidity shock in a region can spread through the interbank market and result in a global crisis. In later chapters we will analyze the model for more general network structures.

3.1 Basic assumptions and preferences

There are two types of players in the world; consumers and banks, and three different time steps, $t = 0, 1, 2$. Consumers are assumed to be identical and in large numbers, each endowed with one unit of a homogeneous consumption good, from now on assumed to be the numéraire. At time $t = 0$, consumers can choose to deposit their endowments in banks, which in turn can invest in assets that yield returns for future consumption. There are two types of assets; short-term and long-term. Short-term assets give a return of 1 unit at time 1. Long-term assets pay a return $r < 1$ if liquidated at time 1 or $R > 1$ at time 2, so it is costly to liquidate the long-term asset at $t = 1$. The banks want to meet demand for liquidity at $t = 1$ by liquidating the short term asset, and demand at $t = 2$ by liquidating the long term asset. Each consumer is either an early consumer, with preference for liquidity at $t = 1$, or a late consumer with preference for liquidity at time $t = 2$. The consumer type is unknown at $t = 0$ and revealed at $t = 1$. At $t = 0$, the expected fraction of early consumers is γ.

At $t = 0$, banks offer consumers a contract that consumers can choose to accept. The market is assumed to be perfectly competitive, such that banks offer the consumers a contract that maximizes the consumers' utility. The utility maximizing allocation is given by (x, y, c_1, c_2) where x is the share invested in long-term asset, y is the share of deposits banks invest in short term assets, c_1 is the amount of consumption offered to consumers that withdraw at $t = 1$, and c_2 the returns to consumers if they withdraw at $t = 2$.

At $t = 1$, consumers decide whether to withdraw their deposits at $t = 1$ or $t = 2$, and banks liquidate assets to meet demand from their depositors. Early consumers always withdraw at $t = 1$. Late consumers withdraw at $t = 1$ or $t = 2$ depending on what gives the most consumption. It is assumed

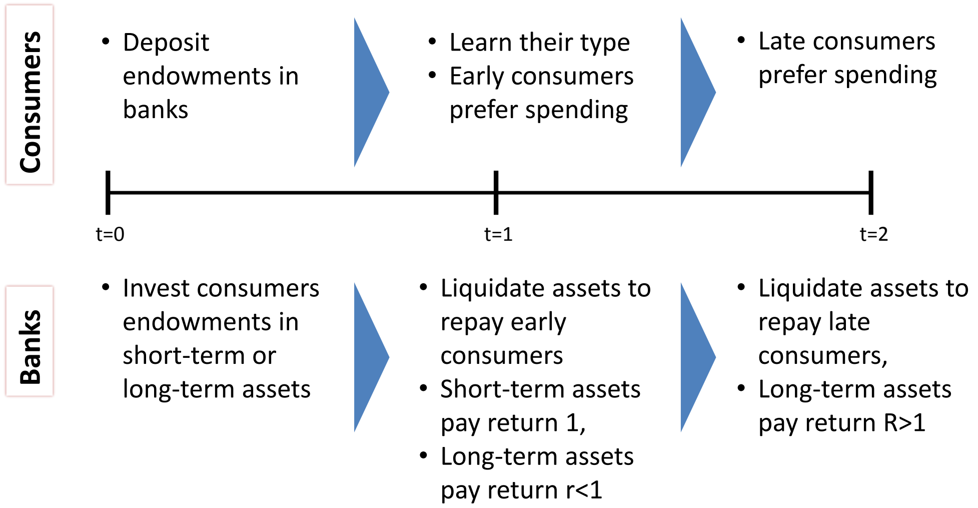


Figure 3.1: Events at different time steps

that late consumers withdraw deposits at $t = 2$ if it is weakly optimal for them, that is if the amount of consumption they get by withdrawing at $t = 2$ is at least as large as what they would get by withdrawing at $t = 1$. If the banks cannot meet the demands of consumers, they must liquidate all assets at $t = 1$. The proceeds are then split equally among the depositors.

Each consumer has the following utility function

$$U(c_1, c_2) = \begin{cases} u(c_1) & \text{with probability } \omega, \\ u(c_2) & \text{with probability } 1 - \omega, \end{cases}$$

where c_t denotes consumption at date $t \in \{1, 2\}$. The utility function $u(\cdot)$ is assumed to be monotone, strictly concave and twice continuously differentiable.

The economy we consider consists of a number of regions. Regions can be interpreted as geographical regions, but could also represent single banks, or a group of financial intermediaries that operate in the same market. The regions are connected by banks' possibility to hold deposits in banks in other regions. All regions face the same optimization problem, and the fraction of early consumers across all regions is known to be γ, so there is no aggregate uncertainty of demand. Regions are in one of two categories; either a region has a higher share of early consumers ω_H, or a lower share of early consumers ω_L. The expected demand γ is thus $\gamma = (\omega_H + \omega_L)/2$.

3.2 Optimal allocation

In the model we assume there is a central planner that seeks to maximize the consumers' utility. The optimization problem faced by the planner is to find the contract (x, y, c_1, c_2) that maximizes the consumers' utility. All consumers in a region are ex-ante identical to consumers in other regions and can be treated alike, such that every early consumer gets c_1 units of consumption and every late consumer receives c_2 units.

The ex ante expected utility for a consumer is $\gamma u(c_1) + (1-\gamma)u(c_2)$. At time $t = 0$ the planner invests in a portfolio $(x, y) \geq 0$ where x and y are the shares of a bank's investments in long term and short term assets respectively. x and y must satisfy the feasibility constraint $x + y \leq 1$. Furthermore, because of the higher returns, but at the same time the cost of premature liquidation associated with the long-term assets, it is optimal to provide for consumption at $t = 1$ by liquidation of the short-term asset, and to provide for consumption at $t = 2$ by liquidation of the long-term asset. Since the expected fraction of early consumers is γ, this can be achieved if the two feasibility constraints $\gamma c_1 \leq y$ and $(1-\gamma)c_2 \leq Rx$ are satisfied.

With the given restrictions on the utility function and under the feasibility constraints, we can formulate the optimization problem faced by the central planner. The problem is a nonlinear, concave maximization problem over a convex domain. The solution is the first-best allocation (x, y, c_1, c_2):

$$\begin{aligned} \underset{x,y,c_1,c_2}{\text{maximize}} \quad & \gamma u(c_1) + (1-\gamma)u(c_2), \\ \text{subject to} \quad & x + y \leq 1, \\ & \gamma c_1 \leq y, \\ & (1-\gamma)c_2 \leq Rx. \end{aligned} \tag{3.1}$$

From monotonicity of u, it follows that all constraints must be binding. We can thus express c_2 in terms of c_1:

$$c_2 = \frac{R(1-\gamma c_1)}{1-\gamma}.$$

We can then solve the first-order optimality condition

$$\frac{du}{dc_1} = 0,$$
$$\gamma u'(c_1) + (1-\gamma)\left(\frac{-R\gamma}{1-\gamma}\right)u'(c_2) = 0,$$
$$u'(c_1) = Ru'(c_2),$$
$$u'(c_1) \geq u'(c_2).$$

The condition $c_1 \leq c_2$ is called the incentive compatibility constraint. We observe that if $c_1 > c_2$, late consumers would get more consumption by withdrawing in period $t = 1$ than in period $t = 2$, so they would prefer to mimic early consumers and withdraw early. By concavity of u, $u'(c_1) \geq u'(c_2)$ implies that $c_1 \leq c_2$. This shows that the incentive-efficient allocation is the same as the first-best allocation.

3.2.1 Interbank market

In Allen and Gale's paper, the market consists of four regions that are ex-ante equal; A, B, C and D. The share of early consumers in a region, ω, varies from region to region. There are two possible values of ω; a higher value, ω_H, and a lower value, ω_L, such that $0 < \omega_L < \omega_H < 1$. The realization of ω in a region depends on the state of nature. There are two states that occur with the same probability, σ_1 and σ_2. The possible realizations are given in table 3.1. We see that although there is regional uncertainty about demand, the correlation pattern of high and low demand regions is known.

	A	B	C	D
σ_1	ω_H	ω_L	ω_H	ω_L
σ_2	ω_L	ω_H	ω_L	ω_H

Table 3.1: States of the world

At $t = 1$ the state of nature is revealed. If a region has a high share ω_H of early consumers, it faces a liquidity deficit of $(\omega_H - \gamma)c_1$ after liquidating the short-term assets. We denote by z the difference between the higher share of early consumers and the expected share of early consumers, such that $z = \omega_H - \gamma$. Similarly, regions with a low share ω_L of early consumers have a liquidity surplus of $(\gamma - \omega_L)c_1 = -(\omega_H - \gamma)c_1 = -zc_1$. There is enough liquidity in the market, but it is inefficiently distributed across regions. For simplicity, when we talk about demand at different time steps, we will for most of the analysis omit c_1 and c_2, and consider the normalized demand, that is the share of consumers that withdraw at a time step.

This maldistribution of liquidity motivates the existence of an interbank market where banks exchange deposits across regions. At time $t = 0$, banks transfer deposits to banks in other regions. Then, at $t = 1$, banks with high demand for liquidity can liquidate holdings in other banks to meet the demand. Banks with low demand keep their deposits in other banks until $t = 2$.

The assumptions we make about bank behavior at the different time steps are thus

1. At $t = 0$ banks place deposits in banks in adjacent regions. The deposits are the minimum possible, such that demand for liquidity can be met at $t = 1$ and $t = 2$ given the following protocol:

2. At $t = 1$ banks liquidate deposits in other regions if necessary to meet demand for liquidity.

3. At $t = 2$ banks liquidate remaining deposits in other regions.

We will see that the behavior described above is consistent with a pareto efficient subgame perfect equilibrium if one of the two states σ_1 or σ_2 is realized at $t = 1$.

Subgame perfect equilibrium

The next step is to specify the game played by banks and to show that the planner's optimal contract can be implemented as a subgame perfect equilibrium. Assume that (x, y, c_1, c_2) is the utility maximizing contract for the consumers proposed by the central planner. Furthermore, perfect competition between banks in a region is assumed, such that the contract offered by a bank is the optimal contract for consumers.

When the contract is accepted and the banks have invested accordingly in short- and long-term assets, banks will seek to ensure that the contract is implemented at subsequent time steps, and consumers will seek to maximize their utility. The consumers and banks then face the following strategic choices:

- At $t = 0$: Banks choose how much to deposit in adjacent regions.
- At $t = 1$: Consumers choose whether to liquidate bank deposits. Banks choose how much of the deposits in other regions to liquidate.
- At $t = 2$: Banks and consumers liquidate all remaining claims.

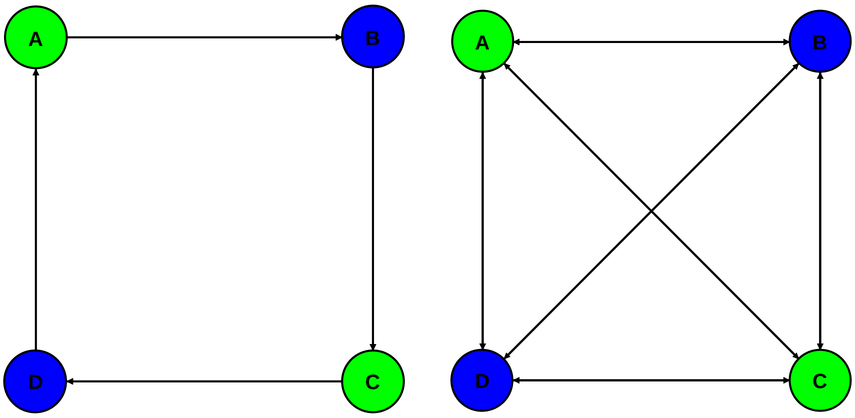
Through these choices, the banks must make sure that consumers who liquidate their deposits at $t = 1$, receive c_1 per unit, and consumers who liquidate at $t = 2$, receive c_2. This game can be solved by backward induction. At $t = 0$, banks place deposits in other banks to be able to meet consumer demand for liquidity at $t = 1$ and $t = 2$, regardless of which of the two states, σ_1 or σ_2, is realized. At $t = 1$, the banks liquidate bank deposits such that they can meet demand at that period, but preferably also at $t = 2$. At $t = 1$, early consumers always withdraw, and late consumers withdraw if and only if it is strictly optimal, that is if it yields strictly higher returns than withdrawing late.

Since there is enough liquidity across regions at $t = 1$ and $t = 2$, assume there exist transfers f and g between regions at $t = 1$ and $t = 2$ respectively, where $f(uv)$ is a transfer from u to v at $t = 1$ and $g(uv)$ is a transfer from u to v at $t = 2$. Assume that w is a set of deposits placed between banks in different regions at $t = 0$ where $w(uv)$ is a deposit placed by region u in region v. Then, if the following conditions hold at the same time for all regions, the choices of w, f and g are strategies in a subgame perfect equilibrium such that the first best optimum is implemented.

1. Transfers f can be achieved through withdrawal of deposits w at $t = 1$, that is $f(uv) \leq w(vu)$.
2. Transfers g can be achieved through withdrawal of deposits w at $t = 2$, that is $g(uv) \leq w(vu)$.
3. For both to be achieved at the same time, we must have $g(uv)+f(uv) \leq w(vu)$.

That this is a subgame perfect equilibrium follows: No bank can implement the first-best contract without transfers from other banks, and since the deposits w achieve the first-best, no bank has incentive to deviate from w. At the subsequent time steps, deficit regions have incentives to withdraw according to the scheme f that ensures that demand can be met. Withdrawals can be enforced, that is a bank cannot keep another bank from liquidating its own deposits, and surplus regions will therefore adhere to the scheme f. The same argument holds for $t = 2$. Therefore, the strategy is a Nash equilibrium in all subgames including the game itself, and we can conclude that it is a subgame perfect equilibrium.

The size and structure of deposits in this equilibrium depends on the graph structure. In later chapters we will consider a variety of graph structures, but we will first repeat the analyses from [2] of two regular graphs; the directed cycle and the complete graph with four nodes.



(a) Directed cycle (b) Complete graph

Figure 3.2: Network structures analyzed by Allen and Gale

Directed circle

We first consider the directed cycle with alternating node types. In the directed cycle every region is connected to only one other region. Early consumer regions are connected to a late consumer region and vice versa.

First we consider an early consumer region. Both the in-neighbor and the out-neighbor are late consumer regions. At $t = 1$, the region needs to meet demand from a share ω_H of consumers in the region. Since the in-neighbor is an early consumer region, it does not want to liquidate its bank deposits, so the total demand is ω_H, and by liquidating the short-term asset the region can obtain γ. The deficit, which it will have to cover by liquidating its deposit in the adjacent region, is $\omega_H - \gamma = z$. The transfer at $t = 0$ to meet the demand at $t = 1$ must be at least z. At $t = 2$, the early consumer region has a surplus of z from liquidating the long term asset. The in-neighbor has a deficit of z, so it will liquidate its deposit, which must be of size z. We can see that demand will be met for both kind of nodes at all times.

Complete graph

The first-best solution in the 4-node complete graph G can be achieved if banks in region u deposit $w(uv) = (\omega_H - \gamma)/2 = z/2$ in all other regions $v \in N_G(u)$. To see this, observe that at time $t = 1$ there will be a liquidity surplus z in the two late consumer regions and an equally sized liquidity deficit in the two early consumer regions. This is the total transfer needed from late consumer regions to early consumer regions to cover the deficit. Each region is connected to two regions of different type than itself, and by depositing $z/2$ at $t = 0$ in each of them, $(\omega_H - \gamma)$ liquidity can be obtained at period $t = 1$ by liquidating deposits in other banks. Note that for the complete graph, transfers between banks in regions of the same type, that is the cross-links in figure 3.2(b), are not needed to achieve the first-best. The same insurance could be obtained from only placing deposits in regions of different type. Regions of the same type face the same demand at the same time, so the mutual claims they hold on each other will cancel out. Allen and Gale, however, assume that all deposits are the same size, and that all links are used. As we will discuss briefly in a later chapter, even if not all edges in a graph are necessary to achieve the first best optimum, including them will generally make the market more resilient towards liquidity shocks.

We see that the deposits each bank must hold in adjacent banks is z for the directed cycle, but only half the size $z/2$ for the complete graph. This difference will play a central role for the robustness of the two market structures when we consider the effect of an unexpected liquidity shock.

Liquidation order

In order to ensure that banks liquidate assets in a certain order, Allen and Gale introduce another condition that induces a "natural" order to liquidate assets:

1. Short-term assets.
2. Bank deposits.
3. Long-term assets.

This particular order is motivated by the cost of liquidating the different assets. A short-term asset is by definition an asset that can be liquidated early, a long-term asset could correspond to building infrastructure, investing in knowledge or similar investments where there the returns are strongly dependent on a longer time horizon, and premature liquidation is costly. A bank deposit should be in between those two. The payoff to investing in the different assets depends on the time perspective, such that short-term assets yield the least returns and long-term assets the most. Having decided the order we want the banks to liquidate their assets, we need to ensure that this is reflected in the model. The short-term asset is worth one unit in period $t = 1$, and one unit if it is reinvested at $t = 2$, so the cost of liquidating is 1. Liquidating the deposit at period $t = 1$ gives c_1 in that period but costs c_2 in future consumption, so the cost is c_2/c_1. Since we want deposits to be liquidated after the short-term asset, we need to have $c_2/c_1 > 1$. We know from the first-order condition that $c_2 > c_1$, so the condition always holds. Furthermore, the long-term asset at $t = 1$ gives r units of consumption, but costs R units of future consumption, so the cost of liquidating early is R/r. The condition $R/r > c_2/c_1$ ensures that the bank liquidates its assets in the desired order.

If a bank goes bankrupt, all the bank's assets are liquidated and all depositors withdraw as much as possible from the bank.

3.3 Liquidity shocks

As long as one of the two states σ_1 or σ_2 is realized, we have seen that the first-best solution can be achieved through an interbank market of deposits. We will now consider the occurrence of unexpected high demand for liquidity in a region at $t = 1$. We introduce a state $\bar{\sigma}$, which is assigned zero probability at time $t = 0$ and has higher total demand for liquidity than the other states. The three states are presented in table 3.2:

	A	B	C	D
σ_1	ω_H	ω_L	ω_H	ω_L
σ_2	ω_L	ω_H	ω_L	ω_H
$\bar{\sigma}$	$\gamma + \epsilon$	γ	γ	γ

Table 3.2: Regional liquidity shock

We see that the average demand for liquidity at $t = 1$ is γ in σ_1 and σ_2, but $\gamma + \epsilon/4$ for an $\epsilon > 0$ in $\bar{\sigma}$. Since $\bar{\sigma}$ is assigned zero probability, the expected demand at $t = 0$ remains γ, and the optimal contract does not change. The equilibrium in states σ_1 and σ_2 will be the same as before.

3.3.1 Equilibrium behavior

It is assumed that banks have full information about the demand for liquidity at $t = 1$. If $\bar{\sigma}$ is realized, the total demand for liquidity in the market cannot be met by liquidating the short-term assets alone. To avoid liquidating the long-term assets, all banks will therefore immediately withdraw all deposits from other banks. Since the claims a bank holds in other regions equals the claims held by banks in other region on that bank, the banks cannot obtain liquidity from other regions, which implies that the regions must be self-sufficient. This will, under certain conditions, lead to bankruptcy in the region where the liquidity shock occurred, and might spread to other regions.

For a modification of Allen and Gale's model, such that banks under certain circumstances can be willing to liquidate some long-term assets before bank deposits to avoid spread of a crisis, see [22].

Buffers and liquidation values

If a bank faces demand for liquidity that cannot be met by liquidating short-term assets and bank deposits alone, the bank is said to be insolvent. If state $\bar{\sigma}$ is realized, it is easy to see that this is the case in the region where the shock occurred. By liquidating short-term assets, the bank can meet a demand of γc_1, but the actual demand is $(\gamma+\epsilon)c_1$. There is thus a shortage of liquidity of ϵc_1. However, insolvency does not necessarily imply bankruptcy.

A bank can afford to liquidate some of the long-term assets prematurely without causing late consumers to withdraw at $t = 1$ and hence cause a bank run. The condition $c_1 \leq c_2$ cannot be violated, that is late consumers must get at least c_1 at $t = 2$, if not they would be better off withdrawing at time $t = 1$. This condition implies that the following inequality must hold:

$$(1 - \omega)c_1 \leq R(x - \beta),$$

where ω is the share of early consumers and β is the amount of long-term assets liquidated at $t = 1$. This gives the maximum amount of long-term assets that can be liquidated prematurely without causing a bank run

$$\beta = x - \frac{(1-\omega)c_1}{R}$$

We can then define the bank's buffer $b(\omega)$, the amount that can be obtained by liquidating the long-term asset at $t = 1$:

$$b(\omega) \equiv r\left(x - \frac{(1-\omega)c_1}{R}\right). \tag{3.2}$$

If the excess demand for liquidity is smaller than the buffer, that is the amount of liquidity that can be obtained by liquidating long-term assets prematurely, late consumers will still prefer to withdraw at $t = 2$ and the bank avoids a bank run. This gives the following criteria for the bank to avoid a run and thus bankruptcy:

$$\epsilon c_1 \leq b(\gamma + \epsilon). \tag{3.3}$$

When this condition is satisfied, the bank itself is able to absorb the liquidity shock. It is insolvent, but not bankrupt, and the other regions will be unaffected. For all of the thesis this condition is assumed violated.

If condition (3.3) is violated, banks in the region will go bankrupt. We assume this is the case in region v. Then all depositors withdraw their holdings at $t = 1$. The bank cannot fully repay the depositors. We denote the value of a unit deposit in the bankrupt bank in region v with q_v, assuming that all depositors split the bank's values equally. The total demand for liquidity is 1 from the consumers and $\sum_{u \in N_G(v)} w(uv)$ from banks in other regions that hold deposits in the region v. The total demand from all depositors is thus $1 + \sum_{u \in N_G(v)} w(uv)$. By liquidating all assets at $t = 1$, banks in region v can obtain liquid means of value $y + rx + \sum_{u \in N_G(v)} w(vu)q_v$ from liquidating short-term assets, long-term assets, and bank deposits. The value of a deposit q_v in v is

$$q_v = \frac{y + rx + \sum_{u \in N_G(v)} w(uv)q_v}{1 + \sum_{u \in N_G(v)} w(vu)}, \tag{3.4}$$

where $q_v < c_1$.

If we assume that no regions other than v are bankrupt, we have $q_v = c_1$ for all $u \in N_G(v)$. This gives an upper bound $\bar{q}_v$ on the value q_v:

$$q_v \leq \bar{q}_v = \frac{y + rx + \sum_{u \in N(v)} w(uv)c_1}{1 + \sum_{u \in N(v)} w(vu)}. \tag{3.5}$$

As we will see in the next chapter, we can find transfers such that $w(uv) = w(vu)$ for all undirected connected graphs.

3.3.2 Spillover effect and contagion

We now consider the situation when state $\bar{\sigma}$ is realized, and region v is subject to a sufficiently large liquidity shock ϵ, such that condition (3.3) is violated. The bank cannot meet the claims from all its depositors and will go bankrupt. The bank needs to liquidate all its assets and deposits.

The value of deposits and assets in the bankrupt region is q_v where $q_v < c_1$. This means that deposits held in the bankrupt bank have lost value, consequently banks that hold deposits in the bankrupt region v will suffer a loss when the deposits are liquidated. This *spillover effect* will make the banks in neighboring regions insolvent, and if the spillover effect is sufficiently large, it can lead to bankruptcy. The spillover effect from a bankrupt region v to its neighbors $u \in N_G(v)$ must be the difference between the claims banks in region v hold on u, $w(vu)c_1$ and the actual value of the claims, $w(vu)q_i$. We can thus find a lower bound on the spillover effect

$$w(vu)(c_1 - q_v) \geq w(vu)(c_1 - \bar{q}_v). \tag{3.6}$$

If the spillover from v to u exceeds the buffer in u, we will have the same effect as when the buffer was exceeded in v; u will face bankruptcy. A necessary condition for banks in region u to avoid bankruptcy when v is bankrupt, must therefore be

$$w(vu)(c_1 - \bar{q}_v) \leq b(\gamma). \tag{3.7}$$

Conversely, if the lower bound exceeds the buffer, we have a sufficient condition to guarantee bankruptcy in region u.

It is easy to see that if u faces bankruptcy, u's neighbors will also be subject to losses caused by the bankruptcies in v and u. If the spillover effects from u and v cause new regions to become bankrupt, we might see a domino effect of a crisis spreading from region to region, and this is what is characterized by Allen and Gale as financial contagion.

Allen and Gale show how this effect plays out in two different graph structures: The directed cycle and the complete graph, and we will see that the complete market is more robust than the directed cycle.

Directed cycle

The first graph structure analyzed by Allen and Gale is the directed cycle. As shown previously, every bank transfers $z = \omega_H - \gamma$ to the adjacent region at $t = 0$. In the four-region network (figure 3.2(a)), A holds a deposit of z in region B, B in C and so on. Assume that the unexpected state $\bar{\sigma}$ is realized, such that A has $\gamma + \epsilon$ early consumers. Furthermore, assume that (3.3) is violated, such that banks in region A cannot avoid bankruptcy. Bankruptcy in region A leads to a loss for region D. If the spillover from A to D exceeds the buffer of D, D will be bankrupt. By using the lower bound on the spillover effect (3.7), we get the following sufficient condition for D to go bankrupt given bankruptcy in A:

$$z(c_1 - \bar{q}_A) > b(\gamma).$$

Since all regions are identical in terms of the size of deposits placed in adjacent banks and their positions in the network, it is easy to show that if the condition above is satisfied, not only will D be bankrupt, but C and B will be bankrupt as well.

In other words, the sufficient conditions to have a global crisis are

$$\epsilon c_1 > b(\gamma + \epsilon) \qquad \rightarrow \text{A bankrupt,} \tag{3.8}$$

$$z(c_1 - \bar{q}_A) > b(\gamma) \qquad \rightarrow \text{Spillover from A exceeds buffer in D,} \tag{3.9}$$

where

$$q_A \leq \bar{q}_A = \frac{y + rx + zc_1}{1 + z}. \tag{3.10}$$

To see that the conditions are sufficient, assume that (3.8) and (3.9) both hold. Then A is bankrupt, and D's deposit in A has lost value. In order to avoid bankruptcy, D must be able to meet the demand for liquidity without exceeding the buffer for liquidation of long-term assets. On the demand side D faces demand γc_1 from consumers and zc_1 from region C. The available liquidity is y from the short-term asset, zq_A from the bank deposit in region A and $b(\gamma)$ from liquidating as much of the long-term asset possible without causing late consumers to withdraw early. Thus the following condition must be satisfied for the bank to avoid bankruptcy:

$$\begin{aligned}(\gamma + z)c_1 &\leq y + b(\gamma) + zq_A \\ &\leq y + b(\gamma) + z\bar{q}_A.\end{aligned}$$

We know from the original optimization problem that $y = \gamma c_1$ is a binding constraint, and therefore holds with equality. From this it follows that the condition to avoid bankruptcy can be rewritten

$$z(c_1 - \bar{q}_A) \leq b(\gamma).$$

We have already assumed (3.9), and the condition to avoid bankruptcy contradicts our assumption, thus D must be bankrupt.

Since regions are identical we have that $q_D \leq \bar{q}_A < c_1$, that is the spillover effect from D to C is at least as large as the spillover effect from A to D. The argument from above can be repeated to show that region C must be bankrupt. When C is bankrupt, the spillover from C will cause bankruptcy in B, and the crisis that initially started in region A has spread contagiously to the whole economy.

Complete graph

The analysis of the complete graph shows the importance of the market structure in spread of a bank crisis. We have found two conditions (3.8) and (3.9) that are sufficient to guarantee a global crisis for the directed cycle if state $\bar{\sigma}$ occurs. It turns out that we can find parameter values that will result in contagion for directed cycle, but where the complete market is able to absorb the shocks, such that there is an equilibrium that does not involve bank runs.

We have already seen that for the complete graph analyzed in [2] a bank holds $(\omega_H - \gamma)/2$ deposits in each adjacent region, that is the claims on each region are smaller, but the total claim held in other regions is higher. We then consider what happens in state $\bar{\sigma}$. By following the same steps as for the directed cycle, we get the following two sufficient criteria for a crisis to be global:

$$\epsilon c_1 > b(\gamma + \epsilon) \qquad \rightarrow \text{Bankruptcy in A.} \tag{3.11}$$

$$\frac{z}{2}(c_1 - \bar{q}_A) > b(\gamma) \qquad \rightarrow \text{Spillover exceeds buffer in B,C,D,} \tag{3.12}$$

where

$$q_A \leq \bar{q}_A = \frac{y + rx + 3zc_1/2}{1 + 3z/2}. \tag{3.13}$$

For the same parameter values r, R, ω_H, ω_L we observe that the following holds

$$z/2\left(c_1 - \frac{y + rx + 3zc_1/2}{1 + 3z/2}\right) < z\left(c_1 - \frac{y + rx + zc_1}{1 + z}\right).$$

It is therefore possible to find a set of parameters, such that condition (3.8) and (3.9) hold, but condition (3.12) is violated. We can thus conclude that the complete market has a better ability than the directed cycle to absorb liquidity shocks and is thus a more robust market structure. The fact that the complete market is more robust can be explained by the fact that each bank is insured against liquidity shocks via several other regions, so the financial interdependencies between regions is smaller.

The main take-away from Allen and Gale is that even small regional perturbations in demand for liquidity can have huge implications for the whole economy. They explain the spread of financial crisis in an interbank deposit market by spillover effects as bankrupt banks' assets lose value. A key feature with their model, is that it points to inherent qualities in the financial market as part of the problem. They also conclude that the complete market is the most robust market structure, and that it generally seems that the more connected the market is the more robust.

3.4 Robustness

The graphs analyzed by Allen and Gale are symmetric. Given that a liquidity shock causes bankruptcy in one region, and that the spillover effect from the bankrupt region to adjacent regions exceeds the buffer, we have a unique symmetric equilibrium where all depositors liquidate their deposits and all banks in all regions face bankruptcy. Robustness will in such cases be defined by the size of the set of parameters that lead to contagion. As soon as we introduce asymmetric graph structures and allow for different sized deposits between banks, regional differences arise, and this complicates the robustness analysis: a) The spread of a crisis can depend critically in which region the initial liquidity shock occurs, and b) a crisis may spread to a nonempty proper subset of the nodes $V' \subset V$ but not to the whole economy. It is therefore not clear how robustness should be defined.

There is some literature on robustness of networks, but few papers focus on financial networks. In the literature robustness is often associated with connectivity properties of the graph, in particular node connectivity. [7] gives an example from communication networks, where the optimal robustness properties are related to optimal connectivity and node similarity.

An interesting contribution to the robustness of financial networks is made in [13], where they are characterized as *robust yet fragile*. The main idea is that financial networks are robust in the sense that the probability of contagion is low; the network has the ability to absorb fairly large shocks. At the same time, if problems occur and the absorption capacity is exceeded, such that a crisis starts spreading, the effect can be extremely large.

Another feature of networks mentioned in the literature, is that networks might be resilient to one type of attack, but not to another. A network that displays robustness towards random attacks, where targeted nodes are chosen at uniform, can at the same time be very vulnerable to a targeted attack on certain key nodes in the network. For an example, see [21].

In this thesis we have chosen to analyze optimal and worst-case situations. In both cases, we have been able to describe the robustness properties in terms of parameter values and the set of bankrupt nodes. For a general analysis, it would be useful with a more precise description of robustness. This is beyond the scope of this thesis, but would be an interesting question to look into for further studies of the Allen and Gale model.

Chapter 4

Framework for robustness analyses

The purpose of this thesis is to analyze robustness properties of different market structures represented by graphs. The analyses will be within the model proposed by Allen and Gale and introduced in the previous chapter.

In the model, banks in different regions are connected through an interbank market of deposits. Banks hold deposits in other regions to insure themselves against uncertainty of demand for liquidity at subsequent time steps. A financial crisis can spread since interbank deposits lose value if a bank goes bankrupt. To analyze robustness of a market structure represented by a graph, we need to consider two steps:

1. Find transfers between banks at $t = 0$ such that the first-best solution can be implemented.

2. Given the transfers, consider the effect of an initial bankruptcy in a region on the rest of the economy.

The first part is important, since larger transfers between banks mean greater financial interdependency, and the effect of a crisis in a region is more likely to effect other regions.

For the analyses, it is useful to formulate the problem in a mathematical framework.

4.1 Notation

The interbank market is represented by the undirected graph $G = (V, E)$. Every region is uniquely represented by a vertex $v \in V$, and there are $n = |V|$ vertices. We say that region u is connected to region v if banks in region u can hold deposits in region v and vice versa. This connection is represented by the edge $e = uv$. The edge set E represents all interbank connections in a market. Only the position in the network gives rise to differences between regions, so the graph G is unlabeled and represents the equivalence class of isomorphic graphs.

There are two disjoint types of regions in the model. The node set V is thus partitioned into two disjoint subsets, V_H and V_L such that $V = V_H \cup V_L$. The subset of nodes V_H represents regions with a higher share, ω_H, of early consumers, and the subset V_L are regions with a lower share, ω_L, of early consumers, where $0 < \omega_L < \omega_H < 1$. It is assumed that the number of early consumer regions equals the number of late consumer regions, so we have $|V_H| = |V_L| = n/2$.

A region v can be connected to early and late consumer regions. In the graph, these are the neighbors $N_G(v)$ of v. We define the set of early consumer nodes in the neighborhood to be

$$N_G^H(v) = \{u : u \in N_G(v),\, u \in V_H\},$$

and similarly for late consumer nodes

$$N_G^L(v) = \{u : u \in N_G(v),\, u \in V_L\}.$$

The number of neigbors is given by the degree of v, denoted by $d(v)$, and the number of early and late consumer regions in the neighborhood are $d^H(v)$ and $d^L(v)$ respectively. We are often interested in the set of edges that connect the two partitions of nodes. Since $V_L = V \backslash V_H$ the boundary is the set of edges between the two subsets $\partial(V_L) = \partial(V_H)$. The number of edges between the subsets is $|\partial(V_H)|$. For shorthand we sometimes use v_H for a node $v \in V_H$ and v_L for a node $v \in V_L$.

Depending on the graph structure and the state of the world, the number of edges through the boundary between V_H and V_L may differ. Since there is a need to transfer liquidity between the different types, it is useful to have a measure of the extent to which different node types are connected. We will talk about *correlation* between node types, where high degree of correlation means that most edges connect nodes that belong to the same node type, whereas low degree of correlation is when there are few edges between nodes of same type. To capture this property of a graph, we define the *correlation coefficient*. We use the simplest and most intuitive approach and define the coefficient as the number of edges that connect nodes of same type divided by the total number of edges.

Definition 4.1. Given a graph $G = (V, E)$ with two disjoint partitions, V_H and V_L, we define the correlation coefficient:

$$Cor(G) = 1 - \frac{|\{e \in E | e = uv,\ u \in V_H,\ v \in V_L\}|}{|E|} = 1 - \frac{|\partial(V_H)|}{|E|}. \tag{4.1}$$

We observe that $Cor(G) = 0$ for bipartite graphs with partitions V_H and V_L. Furthermore, for connected graphs $Cor(G)$ is strictly less than 1.

We denote average demand $\gamma = (\omega_H + \omega_L)/2$ and the difference between average demand and actual demand $z = \omega_H - \gamma = \gamma - \omega_L$. Let $\delta(v)$ denote demand for liquidity in region v. If $\delta(v) > 0$ there is surplus of liquidity in v and if $\delta(v) < 0$ there is a liquidity deficit.

Banks hold deposits in other banks to implement the first-best solution. Since deposits have orientation, it is useful to consider the associated directed graph of G, $D(G)$ with nodeset V and arcset A. Since $D(G)$ is defined from an undirected graph, the set of neighbors in the underlying graph equals the set of in-neighbors equals the set of out-neighbors: $N_D(v) = N_D^+(v) = N_D^-(v)$ for all $v \in V$.

To $D(G)$ we define a function $w : A \to \mathbb{R}$, such that a deposit from banks in region u to banks in region v is represented by $w(uv)$. A feasibility constraint in the economic model is that the sum of the deposits a bank holds in banks in other regions equals the sum of deposits banks in other regions hold in the bank. Recall that we define $w^+(v) := w(\partial^+(v)) = \sum_{a \in \partial^+(v)} w(a)$. We have

$$w^+(v) = w^-(v) \text{ for all } v \in V.$$

We say w is a circulation in $D(G)$. The problem of finding a function w such that the first-best optimum can be achieved, will be called the *circulation problem*.

The motivation for interbank deposits is to be able to meet demand for liquidity at subsequent time steps. The circulation problem will be more precisely defined and we will see how it can be solved in later sections.

4.2 Minimum flow distribution problem

At $t = 1$ and $t = 2$, the banks in a region face either a liquidity surplus or a liquidity deficit, and there is need to transfer liquidity between regions. The problem of redistributing liquidity in the network at these time steps, can be formulated as a network flow problem. Since the problem is to transfer liquidity from a set of sources, nodes with a surplus z of liquidity, to a set of sinks, nodes with a deficit of z, the problem is a multiple sinks and sources problem.

Let $D(G)$ be the associated directed graph of G. We define the network $N := N(X, Y)$ as the digraph D and two particular sets of vertices; a set of sources X, and a set of sinks Y, where $X, Y \subset V$. There are no capacity constraints on the arcs, that is $c(a) = \infty$ for all $a \in A$. Let $f : A \to \mathbb{R}$ be a flow function. If f satisfies the following constraints, we say that f is *feasible*

$$f^+(v) - f^-(v) = 0 \qquad \text{for all } v \in V \backslash (X \cup Y), \tag{4.2}$$
$$f^+(x) - f^-(x) = z \qquad \text{for all } x \in X, \tag{4.3}$$
$$f^+(y) - f^-(y) = -z \qquad \text{for all } y \in Y, \tag{4.4}$$
$$0 \leq f(a) \leq c(a) \qquad \text{for all } a \in A. \tag{4.5}$$

In the problem we analyze, we have $V \backslash (X \cup Y) = \emptyset$, that is all nodes are either sources or sinks. The two constraints 4.2 and 4.5 are trivially satisfied, and will be omitted for the rest of the thesis.

The family of feasible flows f is denoted $\mathcal{F}$. We are interested in finding a feasible f such that the total flow $\sum_{a \in A} f(a)$ is minimized.

$$\underset{\mathcal{F}}{\text{minimize}} \quad \sum_{a \in A} f(a). \tag{4.6}$$

We call this problem the *minimum flow distribution problem.*

In our model, the central planner faces a minimum flow distribution problem at both $t = 1$ and $t = 2$. Let $N_1 := N(X = V_L, Y = V_H)$ be the network with the underlying digraph $D(G)$ at $t = 1$ and let f be the solution to the associated minimum flow distribution problem. Similarly, let $N_2 := N(X = V_H, Y = V_L)$ be the network at $t = 2$ with the same underlying digraph $D(G)$, and let g be the solution to the minimum flow distribution problem in this network. When $D(G)$ is the digraph associated with the undirected graph G, we also say that the minimum flow distribution problem is associated with G.

4.3 First-best allocation

The market is represented by an undirected graph. The first step of the analysis is to find for what graph families it is possible to achieve the first-best allocation. In other words we need conditions to ensure that the problem of distributing liquidity at $t = 1$ and $t = 2$ can be solved. We have seen that the problem can be formulated as the minimum flow distribution problem. We can therefore find conditions under which the problem can be solved.

Theorem 4.1. *Let $G = (V, E)$ be the undirected graph representing the market. The minimum flow distribution problem can be solved for a network $N(X, Y)$ with the underlying digraph $D(G)$ associated to G if and only if for any connected component C in D, we have that $|X(C)| = |Y(C)|$, that is the number of sinks and sources are the same.*

Proof. Let $C = (V(C), E(C))$ be a connected component of $D(G)$. Assume without loss of generality that $|X(C)| > |Y(C)|$. Since C is a connected component, we must have $f^+(V(C)) = f^-(V(C)) = 0$, there is no flow in or out of the component. It follows that the total inflow to nodes must equal the total outflow from nodes.

$$\begin{aligned}
\sum_{x\in X(C)} f^+(x) + \sum_{y\in Y(C)} f^+(y) &= \sum_{x\in X(C)} f^-(x) + \sum_{y\in Y(C)} f^-(y), \\
\sum_{x\in X(C)} f^+(x) - \sum_{x\in X(C)} f^-(x) &= \sum_{y\in Y(C)} f^-(y) - \sum_{y\in Y(C)} f^+(y), \\
\sum_{x\in X(C)} \big(f^+(x) - f^-(x)\big) &= \sum_{y\in Y(C)} \big(f^-(y) - f^+(y)\big), \\
z|X(C)| &= z|Y(C)|.
\end{aligned}$$

This contradicts the assumption that $|X(C)| > |Y(C)|$. We can conclude that $|X(C)| = |Y(C)|$ must hold for every connected component for the problem to have a solution.

The other implication can be shown by induction on nodes. Assume that C is a connected component with two nodes, and $|X(C)| = |Y(C)|$. Let a be the arc with head in Y and tail in X. Then $f(a) = z$ and $f(-a) = -z$ is a feasible flow. Assume that we can find a feasible flow for a connected component C with n nodes and $|X(C)| = |Y(C)| = n/2$. We can then show that we can find a feasible flow for a connected component with $n+2$ nodes, and the induction hypothesis holds. The details are omitted. $\square$

From now on we will assume that all the graphs G we consider are connected.

The next step of the analysis is to find deposits between banks such that the first-best solution can be achieved, that is to find the circulation w associated to the graph $D(G)$.

We show that we can find w by solving the minimal flow distribution problem (4.6) at $t = 1$ with $X = V_L$ and $Y = V_H$.

Proposition 4.2. *Let $G = (V, E)$ be the undirected graph representing the market, and $D(G)$ the associated digraph. Let f be the solution of the minimum flow distribution problem in the network N_1. We can then find functions w and g where w is a solution to the circulation problem and g a solution to the minimum flow distribution problem on N_2, such that the following identities hold*

1. $w(a) = f(-a) + g(-a)$.
2. $f(a) = g(-a)$.
3. $w(a) = f(a) + f(-a)$.
4. $w(a) = w(-a)$.

Proof.
(1) This follows from the economic model, since w represents bank deposits at $t = 0$. Let $a = uv$. Then $f(-a)$ and $g(-a)$ represent bank u withdrawals from region v at $t = 1$ and $t = 2$ respectively. For the withdrawals to be economically feasible, bank u cannot withdraw more than it has deposed $w(a)$ in region v, so we must have

$$w(a) \leq f(-a) + g(-a).$$

In the model banks by definition withdraw all remaining deposits at $t = 2$, and $w(a) \leq f(-a) + g(-a)$ must hold with equality.
(2) We can easily show that $f(a) = g(-a)$ solves the flow distribution problem at $t = 2$. We check that each of the constraints is satisfied. Since $X = V_H$ and $Y = V_L$ at $t = 2$ we have

$$g^+(v_H) - g^-(v_H) = -f^+(v_H) + f^-(v_H) = -(-z) = z,$$

and

$$g^+(v_L) - g^-(v_L) = -f^+(v_L) + f^-(v_L) = -z.$$

Since $v_H \in X$ and $v_L \in Y$, both constraints are satisfied. Let h be any flow that satisfies the feasibility constraints (4.3) and (4.4). Since f is a minimum solution we have

$$\sum_{a \in A} g(-a) = \sum_{a \in A} f(a) \leq \sum_{a \in A} h(a),$$

and we have shown that $f(a) = g(-a)$ solves the minimum flow distribution problem at $t = 2$.
(3) follows directly from (1) on (2): $w(a) = f(-a) + g(-a) = f(a) + f(-a)$.
(4) follows from (3): $w(a) = f(a) + f(-a) = f(-a) + f(a) = w(-a)$. $\square$

The result has two important implications: For any undirected graph we can find a function w of optimal transfers at $t = 0$ such that $w(a) = w(-a)$ and

$$w^+(v) = \sum_{a \in \partial^+(v)} w(a) = \sum_{a \in \partial^+(v)} w(-a) = \sum_{a \in \partial^-(v)} w(a) = w^-(v),$$

which shows that w is a circulation. In economic terms this means that in our model, two adjacent banks always place equally sized deposits in each other. Furthermore we can conclude that in order to find a solution w to the circulation problem at $t = 0$, we can restrict the analyses to finding a solution f to the minimum flow distribution problem at $t = 1$.

4.4 Spread of crisis

The second part of the analysis is to consider the situation when a region in the market is bankrupt. This can also be formulated as a graph theoretic problem. Since we start with a node in the graph $G = (V, E)$ and consider potential spread of a crisis from the node, a breadth first search approach seems natural. The spillover effect from a bankrupt node to another node is directed, hence we will work with the digraph $D(G)$ associated with G.

The market is represented by the digraph $D(G)$ and the spillover effect between regions is represented by a cost function $\mathcal{C} : A \to \mathbb{R}$ associated with each arc. There is a constant capacity function c associated with the node set of the graphs, such that all nodes $v \in V$ have a capacity $c(v) = c$, which represents a region's buffer.

We choose an initial node v in the graph $D(G)$. Node v is removed from the graph (i.e. node is bankrupt). When a node v is removed, it induces a flow, the spillover effect, from v to the neighbors $u \in N_D(v)$. The cost of the flow is given by $\mathcal{C}$. If the cost of the flow exceeds the capacity of node u, the node is removed from the graph. Starting from the initial node v, we can traverse the graph with a breadth first search until no more nodes are removed or the graph is empty. The set of removed nodes B corresponds to the set of bankrupt regions. If $B = V$, the initial crisis has spread to all regions. If $|B| = 1$, that is if it contains only the initial node, the initial regional crisis has not spread at all. The breadth-first search through the graph is presented in algorithm 1.

Algorithm 1 Contagion BFS

Input: Start node v, digraph D, cost $\mathcal{C}$, capacity c
Output: Subset B of nodes V in D

```
Set V = {v}
while V ≠ ∅ do
  Choose v from V
  Add v to B
  for u ∈ N_D(v) but not in B do
    if C(vu) > c then
      Add u to V
    end if
  end for
  Remove v from V
end while
```

Roughly speaking, a graph is considered robust if the set of removed nodes B returned by algorithm 1, is small relative to the full nodeset V.

By considering the algorithm, we see that the size of B depends on the criterion $\mathcal{C}(vu) > c$. If we can show that the criterion is satisfied for all arcs $a = vu$, we have maximum spread, that is $B = V$, and if we can show that it is violated for all arcs $a = vu$, then $|B| = 1$.

The flow cost $\mathcal{C}(a)$ of an arc $a = uv$ is defined to be the spillover effect from region u to region v

$$\mathcal{C}(a) = w(a)\left(c_1 - q_u\right), \tag{4.7}$$

where $w(a) = f(a) + f(-a)$, f is the solution to the minimum flow distribution problem 4.6 and c_1 is the return of a bank deposit at $t = 1$. The unit value of a bank's assets q_u is defined by

$$q_u = \frac{y + rx + \sum_{v \in N_G(u)} w(uv) q_v}{1 + \sum_{v \in N_G(u)} w(uv)}.$$

The capacity c is defined as the buffer $b(\gamma)$ and is constant for all $a \in A$.

Since the main interest is to analyze the effect of the graph structure on robustness, the economic parameters in the model, $\lambda = (\omega_H, \omega_L, r, R)$ will be seen as given, and we will investigate the robustness properties given a set of parameters.

For general graphs we cannot solve the algorithm analytically. It is however possible to derive certain properties of solutions for some graphs by looking at the edge cost structure, which depends on the solution of the multiple source and sink problem, the graph structure, and the economic input parameters in the model.

Redundant edges

Let w be the solution to the circulation problem for an underlying digraph $D(G)$. Consider the arc $a = uv$ in $A(D(G))$. If $w(a) = w(-a) = 0$, we say that the edge $e = uv$ in the edgeset of the associated graph G is redundant in achieving the optimal solution.

The case of redundant edges raises important questions. By using the redundant edges, that is assigning a positive value to $w(a)$ for all a in A, we can obtain a market more resilient towards regional liquidity shocks than we do if we leave the redundant edges out. At the same time, we assume that a central planner seeks to minimize the sum of all deposits banks in different regions hold on each other.

In Allen and Gale, all $w(a)$ are set to the maximum single edge flow, $\max_{a \in A} f(a)$, so also redundant edges are assumed to be used for interbank deposits. The approach in this thesis, where we allow for deposits of different sizes, gives stronger robustness results.

The way we have chosen to treat the problem, is that we discuss properties of the circulation problem and the minimum flow distribution problem without regard to whether all edges in the underlying graph are used or not. For the graphs we have considered, the maximum correlation linear graph, the star, and the k-regular bipartite graph have no redundant edges. The other graphs analyzed have redundant edges.

When we analyze the BFS-algorithm and the cost function $\mathcal{C}$, we assume that all edges have positive flow. For an arc $a = uv$ we define the set of arcs $A(a) := \partial^+(\{u, v\})$. We then define

$$\hat{w}(a) = \min\{w(b) \mid b \in A(a),\ w(b) > 0\}.$$

For $a = uv$ such that $w(a) = 0$, we set

$$\mathcal{C}(a) = \hat{w}(a)\,(c_1 - q_u)\,.$$

In words, for redundant edges in the graph that represents the market structure, the arcs with the same ends in the associated digraph are assigned the minimum positive single value on an arc incident with any of the two nodes connected by the redundant edge. This assumption does not affect the robustness results derived theoretically, as the graphs analyzed have no redundant edges. It is, however, important for simulation.

There are two pragmatic reasons for assigning a positive value to redundant edges. First of all, if we do not use all edges, the network we analyze can end up consisting of several connected components, although the underlying graph that we wanted to analyze is connected. The other reason is that using all edges increases robustness.

The economic rationale behind this assumption can be discussed, and there are many aspects, such as the potential cost of connections, that could alter the logic. Many of the central results we present are independent of this assumption, and there has not been room for a more detailed discussion and analysis of this problem within the frames of this thesis.

Chapter 5

Robustness of graph families

In this chapter, we consider robustness properties of graphs with a fixed number of vertices and edges. In particular, we look at two graph families: Trees, and graphs on $nk/2$ edges where k is a positive integer. The two most important results derived in the chapter are: a) The linear graph with maximum correlation between edges is the least robust network structure, and b) the k-regular bipartite is, in some sense, the most robust network with $nk/2$ edges.

To analyze the robustness properties, we will look to the breadth-first search in algorithm 1 and in particular the criterion $\mathcal{C}(a) > c$. For $a = uv$, the economic interpretation of the condition is that the spillover effect from a bankruptcy in u exceeds the buffer in region v. The condition will be called the *cost-capacity condition*. The cost function $\mathcal{C}$ depends strongly on the minimum flow distribution problem, and it will be useful to derive certain properties of the solution w of the circulation problem. Since w depends only on the solution f of the minimum flow distribution problem, most of the analysis will focus on properties of f. For the rest of this chapter, f is the solution to the minimum flow distribution problem at $t = 1$ with the two nodesets $X = V_L$ and $Y = V_H$ and the underlying graph $G = (V, E)$.

It turns out to be useful to find bounds on the flow f in the graph, since graphs that attain the bounds can be expected to have extremal robustness properties. The robustness results derived in this chapter mostly follow the following procedure:

1. Find upper and lower bounds on total flow, maximum and minimum single edge flow by considering the minimum flow distribution problem.

2. Show that the bounds are attained in the minimum flow distribution problem for certain associated graphs.

3. Use the cost-capacity condition to show robustness results for these graphs.

5.1 Properties of the flow f

In order to be able to derive the results we are interested in for w, we need to establish certain properties of the solution f to the minimum flow distribution problem.

First we observe that since the underlying digraph in the network is associated to an undirected graph, we by definition have that $a \in A$ implies $-a \in A$ for $D(G) = (V, A)$.

Lemma 5.1. *Let f be a solution to the minimum flow distribution problem, and let $a \in A$ be an arc in $D(G)$. Then the following must hold for all a:*

$$f(a) > 0 \Rightarrow f(-a) = 0. \tag{5.1}$$

If there is positive flow along an arc in one direction, there is zero flow along the arc in the opposite direction.

Proof. Since f solves the problem 4.6, the feasibility constraints 4.3 and 4.4 are satisfied. Let $b \in A(D)$ be the arc from u to v. Denote by $\Delta f(b)$ the difference $\Delta f(b) = f(b) - f(-b)$. Let $A' = A \backslash (\{b, -b\})$ be the arcset without b and $-b$. Assume without loss of generality that $f(b) > f(-b)$. The constraints for u are

$$\begin{aligned}
\left|f^+(u) - f^-(u)\right| &= \left|\sum_{a_1 \in \partial_A^+(u)} f(a_1) - \sum_{a_2 \in \partial_A^-(u)} f(a_2)\right|, \\
&= \left|\sum_{a \in \partial_A^+(u)} (f(a) - f(-a))\right|, \\
&= \left|\sum_{a \in \partial_{A'}^+(u)} (f(a) - f(-a))\right| + \Delta f(b) = z.
\end{aligned}$$

This shows that the constraint will be satisfied for any individual values $f(b)$ and $f(-b)$ as long as $\Delta f(b)$ is constant. We can repeat the exact same argument for node v to get that only the difference $\Delta f(b)$ matters. Since $f(a)$ is non-negative for all $a \in A$ and f is the minimum

$$\sum_{a \in A} f(a) = f(b) + f(-b) + \sum_{a \in A'} f(a),$$

we see that the minimum that can be achieved such that the feasibility constraints are still satisfied is when $\Delta f(b) = f(b)$, which implies $f(-b) = 0$. □

Lemma 5.2. *Consider the minimum flow distribution problem on the network $N(X, Y)$. If $a = xy$ where $x \in X$ and $y \in Y$, we have*

$$f(-a) = 0.$$

In other words, there is never positive flow on an arc from a source to a sink.

Proof. We know that at least one of $f(a)$ and $f(-a)$ equals 0. Assume without loss of generality that the net flow through all other edges than a is 0 for both x and y. From the constraints from both x and y we obtain

$$f(a) - f(-a) = z.$$

Since $f(a) \geq 0$ for all $a \in A$ we can conclude that $f(-a) = 0$. □

Lemma 5.3. *The flow $f^+(X)$ out of X equals the flow $f^-(Y)$ in to Y. More specifically,*

$$f^+(X) = f^-(Y) = \frac{nz}{2}.$$

Proof. From conservation of flow we have that for any $V' \subseteq V$ the following holds (see [3])

$$\sum_{v \in V'} (f^+(v) - f^-(v)) = f^+(V') - f^-(V').$$

Since $x \in X$ in our problem satisfies

$$f^+(x) - f^-(x) = z.$$

We get directly that

$$f^+(X) - f^-(X) = z|X| = \frac{zn}{2}.$$

Furthermore, from conservation of flow on the whole network N we have that

$$f^+(X) - f^-(X) = f^-(Y) - f^+(Y) = \frac{zn}{2}.$$

There is no positive flow from Y to X, and we can conclude that

$$f^-(X) = f^+(Y) = 0,$$

and we get

$$f^+(X) = f^-(Y) = \frac{zn}{2}.$$

□

We can then find an upper bound on the total outflow and inflow to a node

Lemma 5.4. *If f is a solution to the minimum flow distribution problem and v a node in the network, we have*

$$f^+(v) \leq \frac{zn}{2},$$
$$f^-(v) \leq \frac{zn}{2}.$$

Proof. The result follow directly from the fact that the sum of net inflow to all nodes in a subset equals the net inflow of the subset. Let X be the set of all sources. We know that for any set $V' \subset V$ the following holds:

$$f^+(V') \leq f^+(X) = \frac{zn}{2}.$$

Assume $v \in X$. Then the maximum possible inflow is $f^-(v) = f^+(X \backslash x) = \frac{zn}{2} - z$. We have $f^+(v) - f^-(v) = z$. We get

$$f^+(v) = z + f^-(v) \leq z + \frac{zn}{2} - z = \frac{zn}{2}.$$

□

5.2 Properties of the circulation w

We will now consider properties of the solution w of the circulation problem. From Lemma 4.2 we know that $w(a) = f(a) + f(-a)$.

Definition 5.1. The following properties of the solution to the circulation problem are defined:

1. *Minimum total flow* is defined as $Q = \sum_{a \in A} w(a)$.
2. *Maximum single edge flow* is defined as $w_{max} = \max_{a \in A} w(a)$.
3. *Minimum single edge flow* is defined as $w_{min} = \min_{a \in A} w(a)$.
4. *Maximum local single edge flow* is defined as $w^v_{max} = \max_{a \in \partial^+(v)} w(a)$.

The motivation for finding bounds on total flow and single edge flow, is that they are central for the robustness analyses later in this chapter: Total flow affects the average case, that is the expected spillover effect from a randomly chosen node. Single edge flow affects worst- and best-case choices of intial nodes in the cost-capacity problem.

First we see that the minimum total flow Q is found directly from the minimum flow distribution problem.

Lemma 5.5. *Let f be a solution to the minimum flow distribution problem. The total flow Q can be written as*

$$Q = 2 \sum_{a \in A} f(a).$$

Proof. By using Lemma 4.2 we get

$$\begin{aligned} Q &= \sum_{a \in A} w(a) \\ &= \sum_{a \in A} (f(a) + f(-a)) \\ &= \sum_{a \in A} f(a) + \sum_{a \in A} f(-a) \\ &= 2 \sum_{a \in A} f(a). \end{aligned}$$

□

A lower bound (LB) on the minimum total flow Q can be found.

Lemma 5.6. *For any graph $G = (V, E)$, the minimum total flow Q is bounded below by zn.*

$$Q \geq zn. \tag{5.2}$$

The lower bound is attained if and only if for a feasible f, the following holds

$$f^-(x) = f^+(y) = 0 \text{ for all } x \in X \text{ and } y \in Y.$$

Proof. Since $Q = 2\sum_{a\in A} f(a)$, it suffices to find a lower bound on $\sum_{a\in A} f(a)$. The flow in the graph $D(G)$ must be conserved, that is the sum of incoming flow over all nodes must equal the sum of outgoing flow from all nodes. Since the flow $f(a)$ contributes to both outgoing and incoming flow, the following must hold

$$\sum_{v\in V}\big(f^+(v) + f^-(v)\big) = 2\sum_{a\in A} f(a).$$

From feasibility constraints in the minimum flow distribution problem, we can deduce the lower bound

$$\begin{aligned}
Q &= 2\sum_{a\in A} f(a)\\
&= \sum_{v\in V}\big(f^+(v) + f^-(v)\big)\\
&= \sum_{x\in X}\big(f^+(x) + f^-(x)\big) + \sum_{y\in Y}\big(f^+(y) + f^-(y)\big)\\
&\geq \sum_{x\in X}\big(f^+(x) - f^-(x)\big) + \sum_{y\in Y}\big(f^-(y) - f^+(y)\big)\\
&= z|X| + z|Y|\\
&= z|V|\\
&= zn.
\end{aligned}$$

We see directly from the calculation that this holds with equality if and only if $f^-(x) = f^+(y) = 0$ for all $x \in X$ and $y \in Y$. □

We can also find an upper bound on the minimum total flow in a graph.

Lemma 5.7. *The minimum total flow Q in any graph $G = (V, E)$ is bounded above by*

$$Q \leq zn^2/2. \tag{5.3}$$

Proof. We consider the minimum flow distribution problem. There is a cost of 1 associated to sending z along an edge. Since we have $n/2$ sinks and $n/2$ sources, we can solve the problem by considering $n/2$ separate single sink and source problems, where the rest of the nodes are considered intermediary nodes. This reduces to an all pairs shortest path problem. To find an upper bound on the solution of the minimum flow distribution problem, it is sufficient to find the worst-case all pairs shortest path solution for the underlying graph $G = (V, E)$. By induction we can show that the maximum all pairs shortest path solution is $n^2/4$ for n even. We first show that it holds for $n = 2$. There is only one pair of nodes, and they are adjacent. This gives a lower bound on the flow of $z2^2/4 = z$, and the induction hypothesis holds for the base case. Assume that the longest shortest past for n nodes is $n^2/4$. Consider the graph with $n+2$ nodes. The longest possible distance between two nodes in a graph on $n+2$ nodes is $n+1$. The maximum all pairs shortest path for the remaining n nodes is $n^2/4$ by the induction hypothesis. The worst case all pairs shortest path on $n+2$ nodes is therefore

$$n + 1 + n^2/4 = \frac{n^2 + 4 + 4n}{4} = \frac{(n+2)^2}{4},$$

and we have shown by induction that $n^2/4$ is the worst-case all pairs shortest path. From this we can conclude that an upper bound on $\sum_{a \in A} f(a)$ must be $zn^2/4$. This gives us the bound on Q

$$Q = 2f(a) \leq 2\left(\frac{zn^2}{4}\right) = \frac{zn^2}{2}.$$

□

Then we find a lower bound on the maximum local single edge flow from a node.

Lemma 5.8. *The largest single edge flow, w^v_{max} from a node v of degree $d(v)$ is at least $z/d(v)$.*

$$w^v_{max} \geq \frac{z}{d(v)}. \tag{5.4}$$

Proof. Let f be the solution of the minimum flow distribution problem. We know that for a node v the following must hold

$$|f^+(v) - f^-(v)| = z,$$

$$|\sum_{a \in \partial^+(v)} f(a) - \sum_{a \in \partial^-(v)} f(a)| = z.$$

Without loss of generality, assume that $\sum_{a\in\partial^+(v)} f(a) > \sum_{a\in\partial^-(v)} f(a)$. We get

$$\sum_{a\in\partial^+(v)} f(a) \geq z$$

Assume for contradiction $f^v_{max} < z/d(v)$

$$\sum_{a\in\partial^+(v)} f(a) \leq f^v_{max} d(v) < \frac{zd(v)}{d(v)} = z.$$

This gives a contradiction, and hence $f^v_{max} \geq z/d(v)$. Since $w(a) = f(a) + f(-a)$ and $f(-a) = 0$ if $f(a) > 0$, we conclude that

$$w^v_{max} = f^v_{max} \geq \frac{z}{d(v)}.$$

□

There is a lower bound on the maximum single edge flow in the graph, w_{max}.

Lemma 5.9. *The maximum single edge flow w_{max} is bounded below by*

$$w_{max} \geq \frac{nz}{2|\partial^+(X)|}.$$

Proof. Consider the minimum flow distribution problem. From Lemma 5.3 we know

$$f^+(X) = f^-(Y) = z|X| = \frac{zn}{2},$$
$$\sum_{a\in\partial^+(X)} f(a) = \frac{zn}{2}.$$

Since X are all sources, this is the largest possible outflow from any subset of nodes in $D(G)$. Assume for contradiction $f_{max} < nz/(2|\partial^+(X)|$.

$$\sum_{a\in\partial^+(X)} f(a) \leq |\partial^+(X)| f_{max} < \frac{zn}{2},$$

which is a contradiction. We conclude that

$$f_{max} \geq \frac{nz}{2|\partial^+(X)|}.$$

Assume a is the arc for which f attains its maximal value $f_{max} = f(a) > 0$. We get

$$w_{max} \geq w(a) = f(a) + f(-a) = f_{max} \geq \frac{nz}{2\partial(X)}.$$

□

Lemma 5.9 indicates how robustness is influenced by the correlation pattern in a graph. The fewer edges in the boundary between X and Y, the fewer edges through which flow can be distributed, which will give larger spillover effects along the edges.

We can also find an upper bound on the total outflow $w^+(v)$ from a node v

Lemma 5.10. *The total outflow $w^+(v)$ of a node v is bounded above by*

$$w^+(v) \leq (n-1)z. \tag{5.5}$$

Proof. Let f be the solution to the minimum flow distribution problem. Assume without loss of generality that $v \in Y$. The upper bounds on inflow and outflow to v are

$$f^-(v) \leq \frac{zn}{2},$$
$$f^+(v) \leq \frac{zn}{2}.$$

The feasible flow f must satisfy the feasibility constraint

$$f^-(v) - f^+(v) = z,$$

which can be written

$$f^+(v) = f^-(v) - z.$$

If we consider $f^+(v) + f^-(v)$ such that the constraint above is satisfied, we get

$$\begin{aligned} f^+(v) + f^-(v) &= f^-(v) - z + f^-(v) \\ &\leq \frac{zn}{2} - z + \frac{zn}{2} \\ &= z(n-1). \end{aligned}$$

We can use this to find a bound on the sum $w^+(v)$

$$
\begin{aligned}
w^+(v) &= \sum_{a\in\partial^+(v)} w(a) \\
&= \sum_{a\in\partial^+(v)} (f(a) + f(-a)) \\
&= \sum_{a\in\partial^+(v)} f(a) + \sum_{a\in\partial^-(v)} f(a) \\
&= f^+(v) + f^-(v) \\
&\leq z(n-1).
\end{aligned}
$$

□

5.3 Trees

The first graph family we will analyze is trees. Trees are minimally connected graphs with $n-1$ edges. By removing any edge, we will disconnect the graph in two connected components.

We will analyze the minimum flow distribution problem for an underlying graph G that is a tree with partitions $X = V_L$ and $Y = V_H$.

Our first result is a lower bound on the minimum single edge flow in a tree.

Lemma 5.11. *If $G = (V, E)$ is a tree with a finite number of nodes, and w a solution to the circulation problem, then $w(a)$ is an integer multiple of z for all a in A, then*

$$w(a) = k(a)z \text{ for all } a \in A,$$

where $k : A \to \mathbb{N}_0$.

Proof. Consider a node v with neighbors $N_G(v)$, and the rooted tree representation of G where v is the root and all neighbors are children. A neighbor and all its descendants form a rooted subtree. If v has $d_G(v)$ neighbors, we can form $d_G(v)$ subtrees. We denote by T_u the subtree of neighbor $u \in N_G(v)$. We will now consider the minimum flow distribution problem on $D(G)$. To simplify notation, we will keep the above-mentioned definitions for the rooted tree representation and the subtrees T_u but with the arc set from $D(G)$ instead of the edge set from G.

For a neighbor u of v, let a be the arc $a = uv$. We know from the proof of lemma 5.3, that the net flow from a subset of node is the sum of the net flow for all nodes in the subset. Therefore, the net flow between u and v must satisfy

$$f(a) - f(-a) = f(T_u),$$

since a and $-a$ are the only arcs connecting T_u to the other nodes in $D(G)$. Assume that $|X(T_u)| = k$ and $|Y(T_u)| = l$. We have

$$f(T_u) = (k - l)z.$$

If $k > l$ we have $f(a) = (k - l)z$ and $f(-a) = 0$, and if $l > k$ we have $f(a) = 0$ and $f(-a) = (l - k)z$. Since k and l are non-negative integers, we can conclude that $f(a)$ or $f(-a)$ is a non-negative multiple of z. Since this holds for an arbitrary choice of v, and any subtree T_u, we can conclude that

$$w(a) = f(a) + f(-a) = |k - l|z \text{ for all } a \in A.$$

□

We will now consider two trees: a) the linear graph, and b) the star graph.

5.3.1 Linear graph

The linear graph is a tree with two vertices of degree 1 (terminal vertices) and the rest of the vertices of degree 2. The robustness properties of the linear graph are sensitive to the correlation pattern between the nodes. We will show robustness results for the minimum and maximum correlation linear graphs.

Minimum correlation linear graph

We consider the linear graph $G = (V, E)$ with alternating node types, that is the bipartite linear graph with bipartitions V_H and V_L.

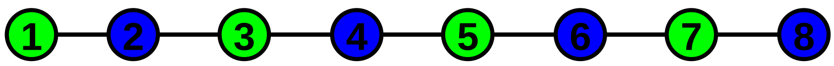


Figure 5.1: Minimum correlation linear graph

Lemma 5.12. *If $G(V, E)$ is a tree on n nodes with the disjoint nodesets V_H and V_L, and we can find a subset $E' \subset E$ such that $M = (V, E')$ is a perfect bipartite matching, then for the associated digraph $D(G)$, there is a subset $A' \subset A$ such that we can find a circulation w for which $w(a) = z$ for all $a \in A'$ and $w(a) = 0$ for all other a. The total flow Q is $Q = zn$.*

Proof. We know that for a tree $w(a) \geq z$ and in general that $Q \geq nz$. If we can find a w such that $w(a) = z$ and $Q = nz$ it must be an optimal solution. Assume G with a perfect bipartite matching $M = (V, E')$. Let $D(M)$ be the associated digraph with arcset A'. There are n nodes, hence the bipartite matching has $n/2$ pairs. Consider the minimum flow distribution problem. Let $a = xy$ and $-a = yx$ be the arcs between a pair in the bipartite matching where $x \in X$ and $y \in Y$. The function f such that $f(a) = z$ for all $a \in A'$ with the tail in X and $f(a) = 0$ for all $a \in A'$ with the tail in Y, and $f(a) = 0$ if $a \notin A'$ satisfies the feasibility constraints. We can then find $w(a)$

$$w(a) = \begin{cases} f(a) + f(-a) = z & \forall\, a \in A', \\ f(a) + f(-a) = 0 & \forall\, a \in A \backslash A'. \end{cases}$$

Since $|A'| = n$, we get

$$Q = \sum_{a \in A} f(a) = |A'|z = nz.$$

□

The result gives the circulation in the minimum correlation linear graph, which is a tree with an underlying parfect bipartite matching.

Since the total flow attains the lower bound, and z is the smallest possible positive value $w(a)$ in a tree, this indicates that a linear graph with an underlying bipartite matching is the optimal linear graph in terms of robustness.

An important question follows from this result. In the solution, some of the edges in the underlying graph are redundant. In fact, the bipartite matching consists of $n/2$ connected components. With this set-up, an initial financial crisis will never spread farther from the origin than the initial region's neighbor in the bipartite matching. For simulations, we have chosen to add flow z to zero-flow edges to ensure that the system we analyze is connected.

Maximum correlation

We will see that the worst-case and most fragile realization of a linear graph is the graph with the maximum correlation coefficient. This is the linear graph $L = (V, E)$ where $|\partial(V_H)| = |\partial(V_L)| = 1$, in other words, there is only one edge connecting the nodesets V_H and V_L.

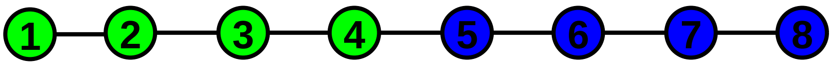


Figure 5.2: Maximum correlation linear graph

Assume $L = (V, E)$ is the maximum correlation linear graph ordered from left to right with nodes $V = \{v_1, v_2, \ldots, v_n\}$ where $v_i \in V_L$ for $i \leq n/2$ and $v_i \in V_H$ for $n/2 < i \leq n$. We want to show that this realization is in some sense the least robust tree graph. First we need to describe w for the network with the underlying digraph $D(L)$.

Lemma 5.13. *If L is the maximum correlation linear graph, and w is the solution to the circulation problem associated to L, we have*

$$w(a) = w(-a) = iz \text{ for } a = v_i v_{i+1} \text{ and } a = v_{n-i} v_{n-i+1},$$

where $i \in [1, \ldots, n/2]$.

Furthermore $w_{max} = zn/2$ and $Q = zn^2/2$.

Proof. Denote by V_i the set of nodes $\{v_j \in V \,:\, 1 \leq j \leq i\}$. Let f be the solution to the minimum flow distribution problem on the network with the underlying digraph $D(L)$. For a node v_i we have

$$f^+(v) - f^-(v) = z = f(\partial(V_{i-1})) + f(\partial(V \backslash V_{i-i}))$$

Assume that $1 \leq i \leq n/2$. Then, by the sum of the net flow of nodes in the subset, we must have $f^+(V_{i-1}) = (i-1)z$. We can therefore conclude that for $a = v_{i-1}v_i$ we have $f(a) = (i-1)z$. We also have $\partial^-(V \backslash V_{i-1}) = iz$, and we can conclude that for $a = v_i v_{i+1}$ we have $f(a) = iz$. It automatically follows that $f(-a) = 0$. This gives the $w(a)$ we wanted to show.

The maximum single edge flow follows directly from Lemma 5.9, since $|\partial(X)| = 1$. The total flow Q is just the sum of $w(a)$, which, from a short calculation can be found to be

$$\sum_{a \in A} w(a) = 2 \sum_{a \in A} f(a) = 2 \times (2(1 + 2 + \ldots + n/2 - 1) + n/2)\, z = \frac{zn^2}{2}$$

□

The next step is to show that both the maximum single edge flow and the total flow are larger than or equal to the optimal transfers in any other tree.

Proposition 5.14. *Let $G = (V, E)$ be any tree and w a solution to the circulation problem. The minimum maximum single edge flow w_{max} satisfies $w_{max} \leq zn/2$ with equality for the maximum correlation linear graph. The total flow Q satisfies $Q \leq zn^2/2$ with equality if and only if G is the maximum correlation linear graph.*

Proof. That the equalities are satisfied for the maximum correlation graph was shown in Lemma 5.13. That $w_{max} \leq zn/2$ follows from Lemma 5.4. The maximum single edge flow is attained for all graphs with $\partial(V_L) = 1$. We need to show that $Q < zn^2/2$ for all other networks than the network with an underlying maximum correlation linear graph. This follows trivially from the fact that the largest possible sum of all pairs shortest paths between pairs of nodes in a graph is $n^2/2$, which can only be achieved for the maximum correlation linear graph. □

We have concluded that the maximum correlation linear graph is the only graph for which the maximum possible minimum total flow and the maximum single edge flow value are achieved in the associated minimum flow distribution problem. This implies that spillover effects are worse than for any other graph both on average, and for best-case and worst-case choice of intial node. We can show that the maximum correlation linear graph is in some sense the worst possible graph in terms of robustness.

Theorem 5.15. *Let G be an undirected connected graph $G = (V, E)$ with n nodes, and let $L = (V', E' =$ be the maximum correlation linear graph with n nodes. Let $B_v(G)$ be the subset of V obtained from algorithm 1 with input $v, D(G), w(G)$ and λ. If*

$$B_v(G) = V \text{ for all } v \in V,$$

then

$$B_u(L) = V' \text{ for all } u \in V'.$$

In economic terms, if for a set of parameters λ and any $v \in V$ we have that bankruptcy in v will spread to all regions in G, the same set of parameters ensure full contagion in L given bankruptcy in any node u in V'.

Proof. Recall that $\mathcal{C}(a) = w(a)(c_1 - q_u)$. From algorithm 1 we see that if $\mathcal{C}(a) > c$ holds for all $a \in A(G)$, we must show that it holds for all $a \in A(L)$. Define

$$\mathcal{C}_{min}(G) = \min_{a \in A(G)} \mathcal{C}(a),$$
$$\mathcal{C}_{min}(L) = \min_{a \in A(L)} \mathcal{C}(a).$$

A sufficient and necessary condition to ensure $B_v(G) = V$ for all v in V is

$$\mathcal{C}_{min}(G) > c,$$

and a sufficient condition to ensure that $B_u(L) = V'$ for all u in V' if $B_v(G) = V$ holds for all v in V is thus

$$\mathcal{C}_{min}(L) \geq \mathcal{C}_{min}(G). \tag{5.6}$$

If we can find an upper bound on $\mathcal{C}_{min}(G)$, and show that $\mathcal{C}_{min}(L)$ is greater than or equal the upper bound, the theorem will be proved. In the minimum flow distribution problem, there is at at least one source $x \in X$ such that $f^-(x) = 0$ and $f^+(x) = z$. Therefore, the maximum minimum value possible $w(a)$ is z. The other property we use is that for $\mathcal{C}(a)$ is decreasing when the sum $w^+(u)$ increases. We can find an upper bound on the $C_{min}(G)$

$$\begin{aligned}\mathcal{C}_{min}(G) &= \min_{a,G} w(a)\left(c_1 - \frac{y + rx + w^+(u)q_u}{1 + w^+(u)}\right)\\ &\leq z\left(c_1 - \frac{y + rx + zc_1}{1 + z}\right),\end{aligned}$$

hence we need to show that

$$\mathcal{C}_{min}(L) \geq z\left(c_1 - \frac{y + rx + zc_1}{1 + z}\right).$$

Observe that for the graph L, node positions i and $n - i + 1$ are identical for $i \in [1, n/2]$. We have seen that $w(a) = iz$ for both $a = v_i v_{i+1}$ and $a = v_{n-i} v_{n-i+1}$, and they also have the same degree $d(v_i) = d(v_{n-i+1})$. When we analyse the function $\mathcal{C}(a)$, we will therefore consider nodes in positions $i \in [1, n/2]$.

A first step is to calculate q_i for each node v_i in the network with the underlying linear graph $D(L)$. We have

$$q_i = \frac{y + rx + (i-1)q_{i-1}z + iq_{i+1}z}{1 + (2i-1)z}. \quad (*)$$

If region i is not bankrupt $q_i = c_1$, and if region i is bankrupt, $q_i < c_1$. We can find an upper bound, $\bar{q}_i$, on q_i by assuming $q_{i-1} = q_{i+1} = c_1$. From $(*)$ we get

$$q_i \leq \bar{q}_i = \frac{y + rx + z + (2i-1)zc_1}{1 + (2i-1)z}.$$

Let a_i be the arc $v_i v_{i+1}$. For a node v_i consider the set of outgoing arcs $\partial^+(v_i) = \{a_i, -a_{i-1}\}$. From lemma 5.13 we can find $\mathcal{C}(a_i)$ and $\mathcal{C}(-a_{i-1})$ respectively

$$\begin{aligned}\mathcal{C}(-a_{i-1}) &= (i-1)z\left(c_1 - q_i\right), \\ \mathcal{C}(a_i) &= iz\left(c_1 - q_i\right).\end{aligned}$$

We see that $\mathcal{C}(a_i) > \mathcal{C}(-a_{i-1})$.

We want to show

$$\mathcal{C}(a_{i+1}) > \mathcal{C}(a_i).$$

To simplify calculations, let $k = 2i - 1$ and $l = 2i + 1$ and observe that $k - l = -2$ and $l - k = 2$.

$$\begin{aligned}
&\mathcal{C}(a_{i+1}) - \mathcal{C}(a_i) = \\
&(i+1)z(c_1 - q_{i+1}) - iz(c_1 - q_i) = \\
&\frac{1}{z}\left(c_1 - (i+1)q_{i+1} + iq_i\right) = \\
&\frac{1}{z}\left(c_1 - (i+1)\frac{y + rx + lzc_1}{1 + lz} + i\frac{y + rx + kzc_1}{1 + kz}\right) = \\
&\frac{1}{z(1+lz)(1+kz)}\left(c_1 + c_1 z(2i-1) - 2zc_1 i - y - rx\right) = \\
&\frac{1}{z(1+lz)(1+kz)}\left(c_1 - y - rx - c_1 z + zy + zrx\right) = \\
&\frac{1}{z(1+lz)(1+kz)}\left((c_1 - y - rx)(1-z)\right).
\end{aligned}$$

From the economic model, we know that $z = \omega_H - \gamma < 1$ and $y + rx < 1 < c_1$. All the factors are positive, and we can conclude that

$$\mathcal{C}(a_{i+1}) > \mathcal{C}(a_i).$$

Let $\bar{\mathcal{C}}(a_i)$ be the upper bound

$$\bar{\mathcal{C}}(a_i) = iz(c_1 - \bar{q}_i) \geq iz(c_1 - q_i) = \mathcal{C}(a_i).$$

We want to show that
$$\mathcal{C}(-a_{i+1}) \geq \bar{\mathcal{C}}(a_i).$$

This is satisfied if the following holds
$$iz(c_1 - q_{i+1}) - iz(c_1 - \bar{q}_i) \geq 0,$$
which implies that we must have $q_{i+1} \leq \bar{q}_i$. This is not trivial, and we need to use additional structure of the problem.

We show $q_{i+1} \leq \bar{q}_i$ by induction on the number of nodes. First consider $n = 2$. From symmetry of the problem, we have that $q_1 = q_2 \leq \bar{q}_1$. Then assume it holds for L on n nodes such that $q_{i+1} \leq \bar{q}_i \; \forall\, i \in [1, n/2]$. Consider the linear graph on $n+2$ nodes. First we observe that node $n/2+1$ and $n/2+2$ are identical, so from symmetry we must have $q_{n/2+1} = q_{n/2+2}$. Since
$$\begin{aligned}
q_{n/2+1} &= \frac{y + rx + (n/2)q_{n/2}z + (n/2+1)q_{n/2+2}z}{1+(n+1)z}, \\
q_{n/2+1} &= \frac{y + rx + (n/2)q_{n/2}z}{1+(n/2)z}, \\
&\leq \frac{y + rx + (n/2)zc_1}{1+(n/2)z} \leq \bar{q}_{n/2},
\end{aligned}$$
the induction hypothesis holds.

For $D(L)$, we have that $\mathcal{C}(a_i) < \mathcal{C}(a_{i+1})$ and that $\bar{\mathcal{C}}(a_i) < \mathcal{C}(-a_{i+1})$. We observe that
$$\begin{aligned}
\mathcal{C}_{min}(L) &= \mathcal{C}(a_1) \\
&= z\left(c_1 - \frac{y + rx + c_1 z}{1+z}\right) \\
&\leq \mathcal{C}(a) \text{ for all } a \in A.
\end{aligned}$$

We then have what we need to conclude the proof:
$$\begin{aligned}
\mathcal{C}_{min}(L) &= z\left(c_1 - \frac{y + rx + c_1 z}{1+z}\right) \\
&\geq \mathcal{C}_{min}(G).
\end{aligned}$$

□

In economic terms, we can conclude that for λ such that $z\left(c_1 - \frac{y+rx+c_1 z}{1+z}\right) > b(\gamma)$, all regions in L will go bankrupt regardless of where a crisis starts. This shows that the maximum correlation linear graph is in some sense the least robust graph.

In the result shown above we have only considered the case of full spread to all regions. The flow properties and degree structure of the maximum correlation linear graph suggest that a more general result holds: For any parameter value there are more nodes from which a crisis will spread, and more regions that will be affected for the maximum correlation linear graph than for any other graph. Establishing such a result seems to require a different approach than what has been used in this thesis, and is therefore left as an open question.

We have seen that the least robust tree structure is the maximum correlation linear graph. We have also shown that a linear graph with an underlying bipartite matching is more robust than any other linear graph. In terms of robustness, however another tree structure does better.

5.3.2 The star

A star is a graph G on n nodes with $n-1$ nodes of degree 1 and 1 node of degree $n-1$. The node set is as usual partitioned into two equally large disjoint subsets V_H and V_L. The star graph has two particularly interesting properties in our model: a) Any partition into two subsets V_H and V_L has the exact same properties, and b) the center node's shock absorbing qualities.

Lemma 5.16. *If G is the star graph and $D(G)$ the associated digraph, we can find the optimal circulation w such that $w(a) = z$ for all $a \in A$ and $Q = 2(n-1)z$*

Proof. Consider the minimum flow distribution problem on the network with the digraph $D(G)$. An arbitrary node in the graph can be in one of two positions: leaf node v_l or the center node v_c. Assume v is a leaf node v_l. A leaf node has degree 1, with only one incoming and one outgoing arc. We get

$$\begin{aligned} |f^+(v_l) - f^-(v_l)| &= z, \\ |f(a) - f(-a)| &= z. \end{aligned}$$

This implies

$$\begin{aligned} f(a) + f(-a) &= z, \\ w(a) &= z. \end{aligned}$$

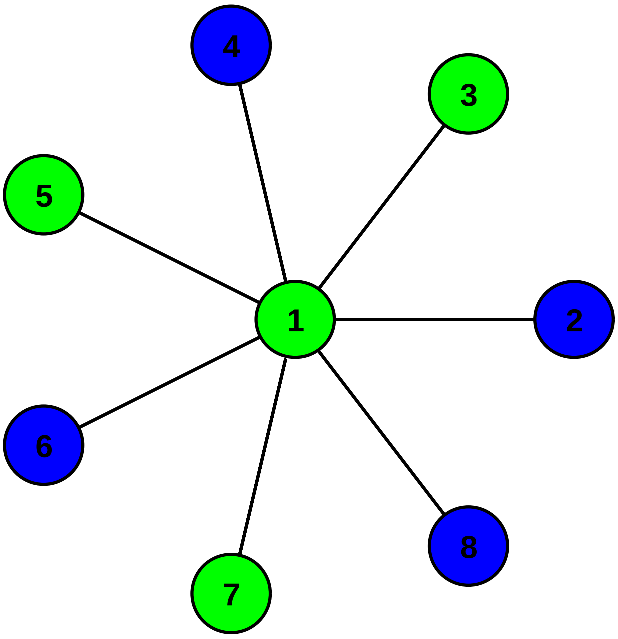


Figure 5.3: Star graph

Since $w(a) = z$ holds for any a with an end in a terminal node, and all a in the star graph has one end in a terminal node and one end in the center node, we can conclude that

$$w(a) = z \text{ for all } a \in A.$$

$\square$

It is easy to see that the spillover effect from the center node in case of bankruptcy is smaller than for any other node in any tree. The center node has thus better shock-absorbing qualities than any other node.

Theorem 5.17. *Let $G = (V, E)$ be the star graph with center node v_c and let $G' = (V, E')$ be any connected tree. Let $A(G)$ be the arcset of $D(G)$ and $A(G')$ the arcset of $D(G')$. Let $a_c \in A$ be any arc from the center node to a leaf node $a = v_c v_l$. The flow $\bar{\mathcal{C}}(a_c) \leq \bar{\mathcal{C}}(a)$ for any $a_c \in A$ and $a \in A'$.*

Proof. Consider $a = uv \in A'$. The cost $\mathcal{C}(a)$ is bounded above by

$$\bar{\mathcal{C}}(a) = w(a)\,(c_1 - \bar{q}_u)$$

We know that for any tree G' we have $w(a) \geq z$ and that $w^+(u) \leq (n-1)z$. From Lemma 5.16 we know that for the star graph G we have $w(a) = z$ and consequently $w^+(v_c) = (n-1)z$. This gives

$$\begin{aligned}
\bar{\mathcal{C}}(a) &= w_a\,(c_1 - \bar{q}_u) \\
&\geq z\,(c_1 - \bar{q}_u) \\
&= z\left(c_1 - \frac{y + rx + w^+(u)c_1}{1 + w^+(u)}\right) \\
&\geq z\left(c_1 - \frac{y + rx + (n-1)zc_1}{1 + (n-1)z}\right) \\
&= z\,(c_1 - \bar{q}_c) \\
&= \bar{\mathcal{C}}(a_c).
\end{aligned}$$

□

5.4 Graphs with $nk/2$ edges

We will now consider graphs with $nk/2$ edges, that is graphs with an average degree k where k is a positive integer. The main result in this section is a robustness result for the k-regular bipartite graph.

5.4.1 The k-regular bipartite graph

We will show that for n nodes and $kn/2$ edges, the bipartite k-regular graph is the most robust graph structure.

The first step is to find the optimal circulation w for the k-regular bipartite graph. It can be shown that the optimal transfers are z/k through all edges.

Lemma 5.18. *Let $G = (V, E)$ be the k-regular bipartite graph with bipartitions V_H and V_L. We can find a circulation w such that $w(a) = z/k$ for all $a \in A$. The total flow Q in the graph is $Q = nz$.*

Proof. We first observe that the graph is symmetric, so we must have a symmetric equilibrium such that $f(a)$ is identical for all $a \in A$. Each node v has outdegree and indegree k. We consider the minimum flow distribution

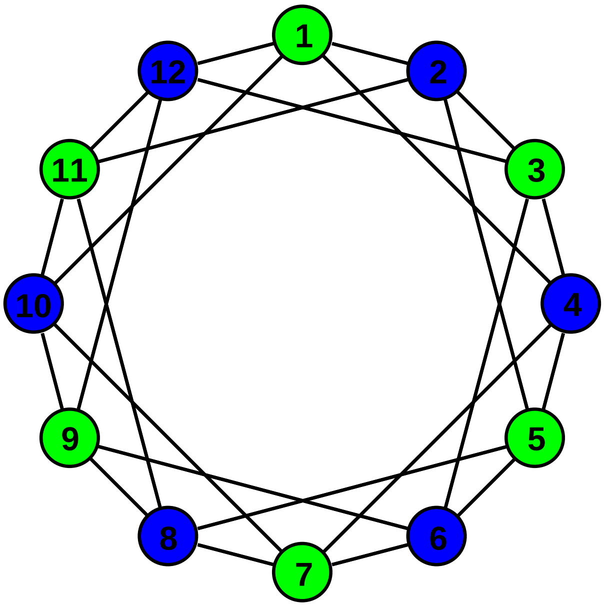


Figure 5.4: 4-regular bipartite graph

problem. For a node $v \in V$ we have that

$$\begin{aligned} |f^+(v) - f^-(v)| &= z, \\ k|f(a) - f(-a)| &= z, \\ k(f(a) + f(-a)) &= z, \\ f(a) + f(-a) &= \frac{z}{k}, \\ w(a) &= \frac{z}{k}, \end{aligned}$$

which is the result we wanted to show. Summing over all edges we get

$$Q = \sum_{a \in A} w(a) = nk \times \frac{z}{k} = nz.$$

□

We can then show that the following holds

Lemma 5.19. *Let $G = (V, E)$ is a graph with $|V| = n$ and $|E| = nk/2$ and w the solution to the circulation problem for $D(G)$, the following two statements are equivalent:*

1. *Maximum single edge flow is* $w_{max} = z/k$.
2. *G is the k-regular bipartite graph.*

Proof. We have already shown in the previous lemma that the second implies the first. What is left to prove is that the regular bipartite graph is the only graph G where the lower bound for w_{max} is attained. Assume the graph G is not regular. Since the average degree is k, there must be at least one node v with degree $d(v) < k$. Then from Lemma 5.4 we get

$$w^v_{max} \geq z/d(v) > z/k.$$

Regularity is therefore necessary for $w_{max} = z/k$ to hold. Assume G is regular but not bipartite. Then there is at least one edge between nodes of same type. Consequently, if we consider the minimum flow problem on $D(G)$, we observe

$$\partial^+(X) < \frac{nk}{2}.$$

By using Lemma 5.9 we get

$$w_{max} \geq \frac{nz}{2|\partial^+(X)|} > \frac{2nz}{2nk} = \frac{z}{k}.$$

We can conclude that $w_{max} = z/k$ if and only if G is the k-regular bipartite graph. □

The next step, is to show that we can find parameter values such that the spillover effect causes bankruptcies for all graphs but the k-regular bipartite graph. We can show that we can find parameter values such that it holds for a non-empty subset of nodes.

Theorem 5.20. *Let* $G = (V, E)$ *be the* k *regular bipartite graph and* $G' = (V, E')$ *graph with the same nodeset that is not the* k*-regular bipartite graph. Then there exist parameter values* $\lambda = (r, R, \omega_H, \omega_L)$ *such that* $|B_v(G)| = 1$ *for all* $v \in V$*, but* $|B_v(G')| > 1$ *for a nonempty subset of nodes* $v \in V$.

Proof. For G the condition to ensure $|B_v(G)| = 1$ is

$$\bar{\mathcal{C}}(a) = \frac{z}{k}(c_1 - \bar{q}) = \frac{z}{k}\left(c_1 - \frac{y + rx + zc_1}{1+z}\right) \leq c. \tag{5.7}$$

Let λ be the parameter values and G the k-regular graph such that

$$\bar{\mathcal{C}}(a) = c. \tag{5.8}$$

For G' we have seen that $w_{max} > z/k$. Assume without loss of generality that $w_{max} = w(a)$ for $a = uv$. Consider $\mathcal{C}(a)$. Assume that u has degree $d(u)$. Then $w_{max} \geq z/d(u)$. This gives

$$\begin{aligned}
&w_{max}\left(c_1 - \frac{y + rx + w^+(u)c_1}{1 + w^+(u)}\right) \geq \\
&w_{max}\left(c_1 - \frac{y + rx + d(u)w_{max}c_1}{1 + d(u)w_{max}}\right) \geq \\
&w_{max}\left(c_1 - \frac{y + rx + zc_1}{1 + z}\right) > \\
&\frac{z}{k}\left(c_1 - \frac{y + rx + zc_1}{1 + z}\right).
\end{aligned}$$

From this we can conclude that along the edges with the largest transfer, w_{max}, the spillover will exceed the buffer for any other graph except the k-regular bipartite graph given that (5.8) holds. In fact this holds for all nodes with incident edges with $w > z/k$. □

This shows that the k-regular bipartite graph absorbs shocks better than any other graph with $nk/2$ edges, both on average and in worst-case. We may, however, still be able to find graphs G such that an initial bankruptcy will cause full contagion in the k-regular bipartite graph, but where there might be nodes v in G from which the spillover does not exceed the buffer in adjacent regions.

5.4.2 The minimum correlation lattice

The minimum correlation lattice does generally not have extremal properties like the k-regular bipartite graph. Since it can be seen as a generalization of the complete market considered by Allen and Gale, it is of interest to derive certain properties.

Definition 5.2. The k regular ring lattice is the graph $G = (V, E)$ where, if we label the edges $V = (v_0, v_1, \ldots, v_{n-1})$, there is an edge (v_i, v_j) if and only if $|i - j| \equiv k \mod n$ for $k \in [1, k/2]$.

The complete graph is thus a lattice with $k = n - 1$.

We then consider a lattice wher early and late consumer regions are organised in alternating order. Let $G = (V, E)$ be a k-regular ring lattice with nodes

$\{v_1, v_2, \ldots, v_n\}$. We define the minimum correlation as the lattice where $v_i \in V_H$ for odd i and $v_i \in V_L$ for even i. The number of neighbors each node has of each node type is

$$d^H(v_H) = d^L(v_L) = 2\lfloor k/4 \rfloor,$$
$$d^H(v_L) = d^L(v_H) = 2\left(\lfloor (k-1)/4 \rfloor + 1\right).$$

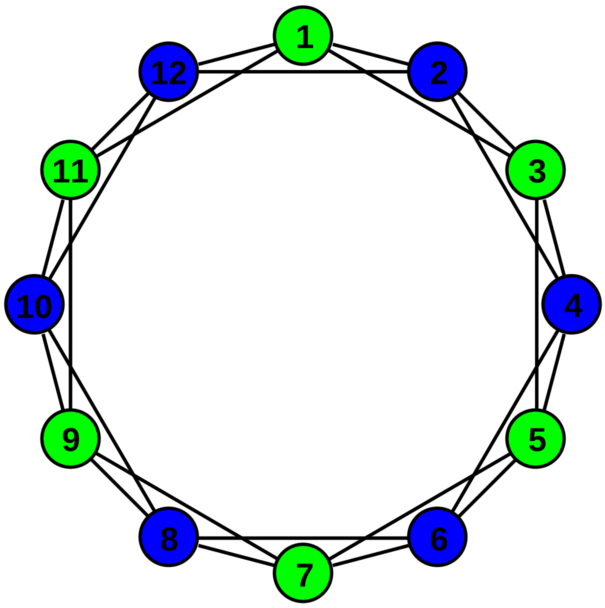


Figure 5.5: Minimum correlation 4-lattice

Lemma 5.21. *Let $G = (V, E)$ is the k-regular minimum correlation lattice. We can then find a solution w to the circulation problem associated wtih $D(G)$ such that*

$$w(a) = \frac{z}{d^H(v_L)} \qquad \textit{for } a \textit{ in } \partial(V_H),$$
$$w(a) = 0 \qquad \textit{for all other } a \textit{ in } A.$$

Proof. Let f be a flow function associated with $D(G)$. We have that

$$f(a) = \frac{z}{d^H(v_L)} \qquad \text{for } a \text{ in } \partial^+(X),$$
$$f(a) = 0 \qquad \text{for all other } a \text{ in } A,$$

where f a feasible flow that satisfies the minimum flow distribution problem associated with $D(G)$. That the flow is feasible follows directly from insertion in to the feasibility constraints. That the flow is a minimum flow, and therefore satisfies the minimum flow distribution problem follows by Lemma 5.2

$$\sum_{a \in A} f(a) = \frac{z}{d^H(v_L)} |\partial^+(X)| = \frac{z}{d^H(v_L)} \times \frac{nd^H(v_L)}{2} = \frac{zn}{2}.$$

From Proposition 4.2 we know that $w(a) = f(a) + f(-a)$, and we get the result we want. □

We see that the edges between nodes of the same type are redundant in solving the minimum flow distribution problem for the k-lattice. In Allen and Gale, the redundant edges in the complete graph, a special case of the k-lattice with $k = n-1$, are assigned the same value as the other edges. This is also done in numerical simulations in this thesis.

Chapter 6

Numerical examples

We have found robustness results for certain regular graph families. For most graphs, however, it is impossible to solve the model analytically because of the complexity of the problem. This motivates numerical simulations of the model. We will illustrate the theoretical results from the previous chapter through simulations for various parameter values and graph structures, and we will compare the equilibrium solutions of different graphs with a fixed number of edges.

6.1 Parameters and input values

In the model and with the assumptions presented in this thesis, the problem is defined by the following exogeneous parameters: The economic parameters $\lambda = (r, R, \omega_H, \omega_L)$, where r and R are the liquidation values of a long-term asset at $t = 1$ and $t = 2$ respectively, the network structure G, the node correlation pattern σ, and the region of the initial shock v. We also need to choose a utility function u that is monotone, concave and twice continously differentiable.

In [8], Douglas W. Diamond analyzes a numerical example of the Diamond-Dybvig model, that Allen and Gale's model build on. Diamond's choice of utility function is $u(c) = 1 - 1/c$. This utility function will be used in all numerical examples in this chapter.

The different parameters influence the problem in different ways. The purpose of this thesis, and what we have discussed in previous chapters, is mainly to discuss how the network structure and correlation pattern influence the robustness of the system. In order to simulate the problem, we need to decide how to choose the other parameters involved.

The demand at $t = 1$ in high and low demand regions, ω_H and ω_L, is kept fixed throughout all simulations. This could be any two values. In [8], the demand ω in a region is 0.25, so a reasonable choice may be $\omega_H = 0.35$ and $\omega_L = 0.15$. This gives $\gamma = 0.25$ and $z = 0.1$. For the return, R, of the illiquid, long-term asset, we also follow Dybvig and choose $R = 2$.

For the choice of r, observe that r does not affect the first-best allocation or the interbank deposit size. Since r represents the value of the illiquid asset if prematurely liquidated, it only affects the outcome in the case of a liquidity shock. The smaller r the bigger is the loss of value from premature liquidation, which will affect the buffer and the spillover effect. Keeping all other parameters fixed, we can use r to "tune" the model such that we get spread of crisis. The first step is to write the spillover condition as a function, Φ, where $\Phi > 0$ implies that the spillover exceeds the buffer:

$$\Phi = w(uv)(c_1 - \bar{q}_u) - b(\gamma). \tag{6.1}$$

We can then substitute for $\bar{q}_u$ and $b(\gamma)$ to get

$$\Phi = w(uv)\left(c_1 - \frac{y + rx + \sum w(uv)c_1}{1 + \sum w(uv)}\right) - r\left(x - \frac{(1-\gamma)c_1}{R}\right).$$

Take the partial derivative with respect to r

$$\begin{aligned}
\frac{\partial \Phi}{\partial r} &= \frac{-w(uv)x}{1 + \sum w(uv)} - \left(x - \frac{(1-\gamma)c_1}{R}\right) \\
&= \frac{(1-\gamma)c_1}{R} - \frac{w(uv)x}{1 + \sum w(uv)} - x \\
&= \frac{Rxc_1}{Rc_2} - \frac{w(uv)x}{1 + \sum w(uv)} - x \\
&= x\frac{c_1}{c_2} - x - \frac{w(uv)x}{1 + \sum w(uv)} \\
&= \left(\frac{c_1}{c_2} - 1\right)x - \frac{w(uv)x}{1 + \sum w(uv)} < 0.
\end{aligned}$$

The last inequality follows from the fact that $c_2 > c_1$ and $x > 0$ and all $w(uv)$ non-negative. Thus we can conclude that there is a threshold such that for r-values below the threshold, condition 2 will hold, and for r-values above the threshold the condition will be violated. This threshold can easily be found as a function of the optimal contract, the node degree and the size of the deposit, $w(uv)$. The threshold is when $\Phi = 0$, so a simple calculation gives an explicit expression for the threshold:

$$r_{threshold} = \frac{w(uv)c_1 - w(uv)\frac{y+\sum w(uv)c_1}{1+\sum w(uv)}}{\frac{w(uv)x}{1+\sum w(uv)} + x - \frac{(1-\gamma)c_1}{R}}.$$

For a given set of input parameters and graph structure, we have that for $r > r_{threshold}$ the spillover effect from a region will not exceed the buffer of another region given bankruptcy in the inital region. We denote by r_{max} the maximum value for r for which an intitial bankruptcy might spread to a nonempty nodeset, that is for r_{max} there is at least one node i and one node j such that a bankruptcy in i will cause the spillover from i to j to exceed the buffer and cause bankruptcy in j. On the other end of the scale, we denote by r_{min} the maximum value for r such that the spillover is exceeded between any two nodes (the maximum minimum value). A high r_{min} will indicate a vulnerable graph structure, and a low value of r_{max} will indicate a robust graph structure.

The optimal contract

The optimal contract is independent of the graph structure. Since the optimal contract is affected only by the utility function $u(\cdot)$ and the parameters ω_H, ω_L, R, we can find the first-best allocation that will be used for all simulations.

Solving the original optimization problem (3.1) and finding the first best allocation is trivial, and implementation is straightforward using standard Matlab functions.

From Chapter 3 we know that the optimal solution of the problem is the solution of the following equation

$$\gamma u'(c_1) - R\gamma u'\left(\frac{R(1-\gamma c_1)}{1-\gamma}\right) = 0.$$

By inserting for the chosen parameter values and solving the equation numerically, we find the following optimal allocation (x, y, c_1, c_2) for the parameters

$\lambda = (\omega_H, \omega_L, R)$

$$\begin{pmatrix} \omega_H = 0.35 \\ \omega_L = 0.15 \\ R = 2 \end{pmatrix} \Rightarrow \begin{pmatrix} x = 0.68 \\ y = 0.32 \\ c_1 = 1.28 \\ c_2 = 1.81 \end{pmatrix}. \tag{6.2}$$

This means that for the given input parameters, the banks will invest 68% of the portfolio in long term assets and 32% in short term assets. The returns to consumers at $t = 1$ is 1.28 per unit and 1.81 per unit at $t = 2$.

Initial liquidity shock

Like before, we are interested in analyzing equilibrium in state $\bar{\sigma}$. Furthermore we require the initial region to be bankrupt. In the model, the size of the liquditidy shock ϵ is exogenously given. To ensure bankruptcy we choose a large ϵ such that bankruptcy in the initial region always is ensured. Given γ, r, R the minimum ϵ that would cause bankruptcy in the region of the initial shock can be calculated. Since we use r as a tuning parameter, and the minimum ϵ depends on r, which affects the buffer size, we use r_{min} to calculate the minimum ϵ, and it will therefore vary with the graph structure.

Market structure

We will consider the same graph families that have been analyzed in the previous chapter, namely graphs on $n-1$ edges and $nk/2$ edges. All simulations will be for $n = 20$. For graphs on $n - 1$ edges, simulations are done for the maximum and minimum correlation linear graphs, the star and a random tree. The structures with $nk/2$ edges that we simulate are the k-regular bipartite graph, the k-lattice with maximum and minimum correlation, as well as a random graph generated with the standard Watts and Strogatz procedure with rewiring probability 1.

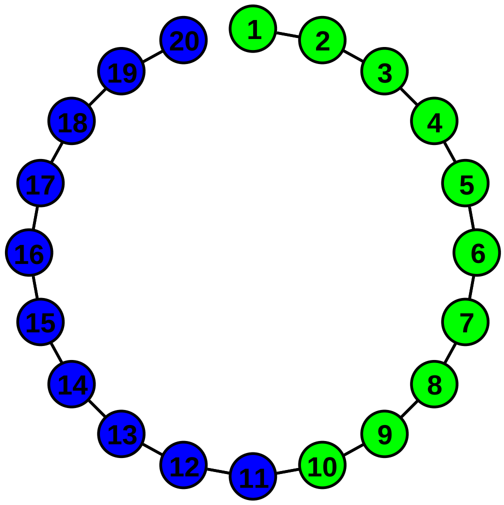
6.2 Trees

Trees are generally the most fragile connected graphs, since they are minimally connected with less edges than any other connected graph. In this section we present results from simulations as explained in the previous chapter.

We will see resuls for the maximum and minimum correlation linear graphs, the star, and a randomly generated tree. The results turn out to be coherent with the theoretical results presented in the previous chapter.

6.2.1 Maximum correlation linear graph

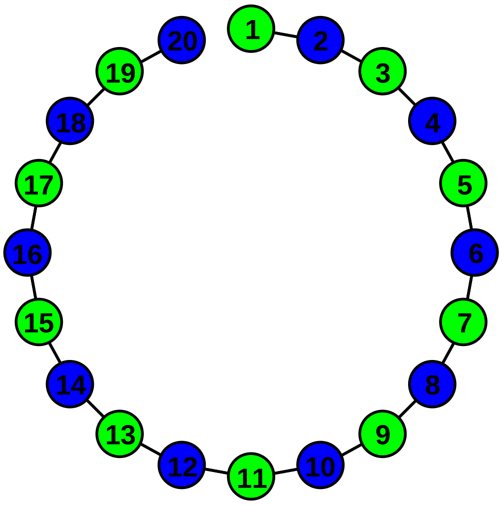
We first present the results from simulation for the maximum correlation linear graph. From the previous chapter it is known to be the least robust market structure, hence one can expect r_{max} and r_{min} to be larger than for any other graph.



Maximum correlation linear graph	
Diameter	19
Clustering coefficient	0
Correlation coefficient	0.947
Maximum transfer ($\times z$)	10
Minimum transfer ($\times z$)	1
Maximum degree	2
Minimum degree	1
Buffer size	0.059
Minimum ϵ	0.054
r_{min}	0.335
r_{max}	0.765

6.2.2 Minimum correlation linear graph

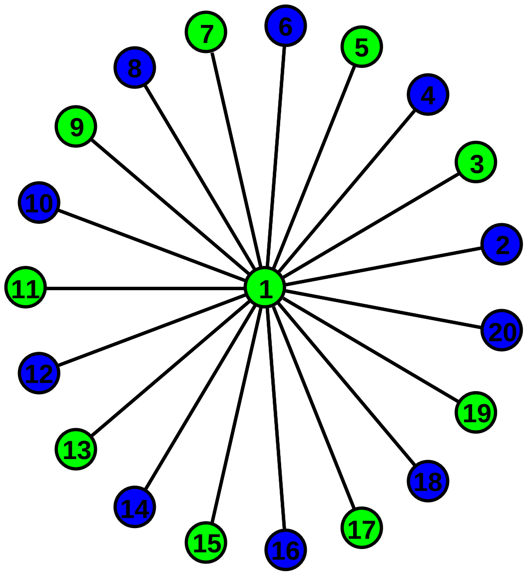
The simulation confirm what is known from the previous chapter; that the transfers are z along all edges, and that the end nodes are slightly more vulnerable towards liquidity shocks than other nodes, since there is only one adjacent node.



Minimum correlation linear graph	
Diameter	19
Clustering coefficient	0
Correlation coefficient	0
Maximum transfer ($\times z$)	1
Minimum transfer ($\times z$)	1
Maximum degree	2
Minimum degree	1
Buffer size	0.062
Minimum ϵ	0.058
r_{min}	0.313
r_{max}	0.335

6.2.3 The star

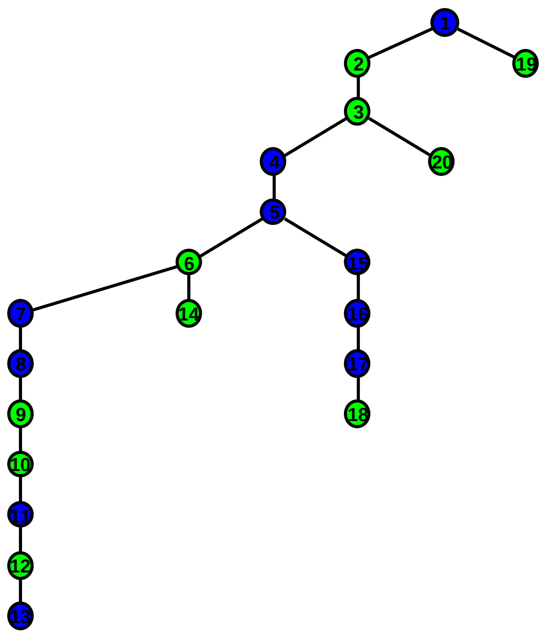
The star graph is of interest mostly for the shock absorbing properties of the center node, which in the simulation is reflected by the low r_{min}.



Star graph	
Diameter	2
Clustering coefficient	0
Correlation coefficient	0.47
Maximum transfer ($\times z$)	1
Minimum transfer ($\times z$)	1
Maximum degree	19
Minimum degree	1
Buffer size	0.030
Minimum ϵ	0.025
r_{min}	0.149
r_{max}	0.335

6.2.4 Random tree

The random tree is included in the simulation to give an understanding of the dynamics of the model. The tree was generated by starting with an empty graph, and then until all nodes were connected, repeatedly choosing two nodes at random and adding an edge between them if this did not create a cycle.



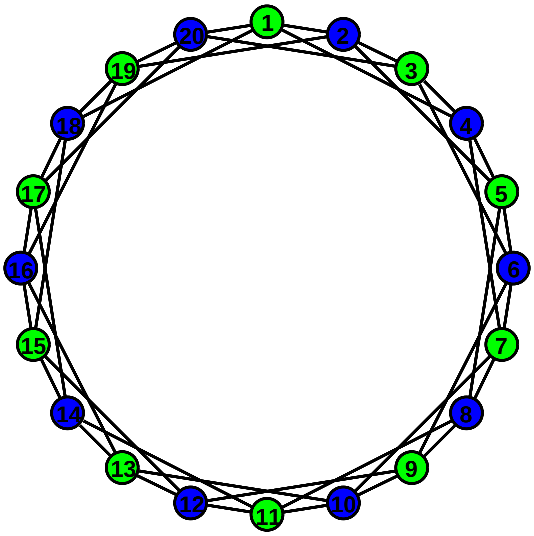
Random tree	
Diameter	13
Clustering coefficient	0
Correlation coefficient	0.47
Maximum transfer ($\times z$)	3
Minimum transfer ($\times z$)	1
Maximum degree	19
Minimum degree	1
Buffer size	0.052
Minimum ϵ	0.047
r_{min}	0.26
r_{max}	0.57

6.3 Graphs with $nk/2$ edges

In the simulation, we have considered graphs with 20 nodes and 40 edges. For this family of graphs, the k-regular bipartite graph, is shown to be the most robust. We also consider the minimum and maximum correlation k-lattice, and a random graph.

6.3.1 The 4-regular bipartite graph

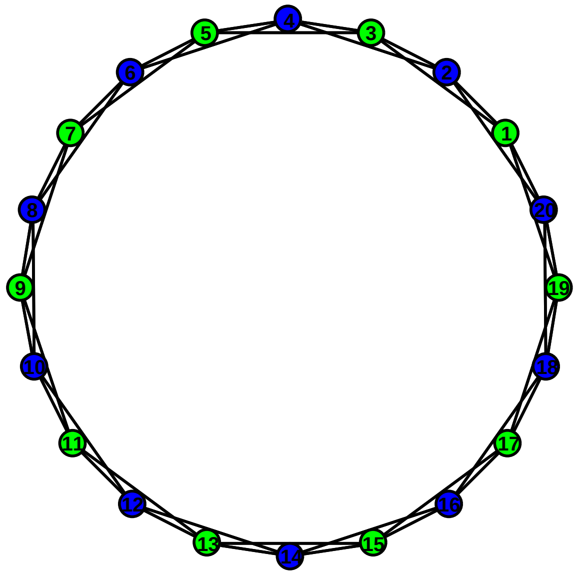
Since the k-regular graph is the most robust graph with $nk/2$ edges, we expect r_{max} to be lower than for all other graphs with the same number of nodes and edges. This is coherent with the results from the simulations



4-regular bipartite graph	
Diameter	4
Clustering coefficient	0
Correlation coefficient	0
Maximum transfer ($\times z$)	0.25
Minimum transfer ($\times z$)	0.25
Maximum degree	4
Minimum degree	4
Buffer size	0.020
Minimum ϵ	0.017
r_{min}	0.102
r_{max}	0.102

6.3.2 The minimum correlation lattice

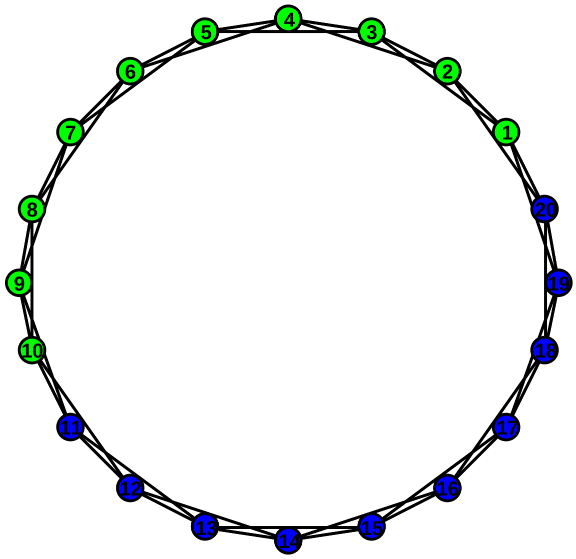
This graph is included, since it can be seen as a generalization of the complete graph analyzed by Allen and Gale. The results from the simulation reflects the theoretical results from the previous chapter.



Min.corr. 4-lattice	
Diameter	5
Clustering coefficient	0.5
Correlation coefficient	0.5
Maximum transfer ($\times z$)	0.5
Minimum transfer ($\times z$)	0.5
Maximum degree	4
Minimum degree	4
Buffer size	0.035
Minimum ϵ	0.030
r_{min}	0.176
r_{max}	0.176

6.3.3 The maximum correlation lattice

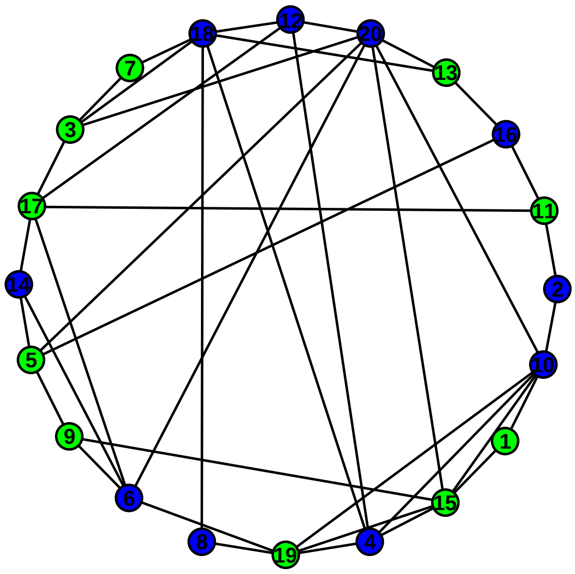
The maximum correlation lattice is included for comparison with the minimum correlation lattice to illustrate the importance of the correlation patterns. As we can see, r_{max} is higher than for the minimum correlation lattice.



Max.corr. 4-lattice	
Diameter	5
Clustering coefficient	0.5
Correlation coefficient	0.85
Maximum transfer ($\times z$)	2.4
Minimum transfer ($\times z$)	0.1
Maximum degree	4
Minimum degree	4
Buffer size	0.009
Minimum ϵ	0.007
r_{min}	0.043
r_{max}	0.522

6.3.4 Random graph

The random graph is generated by applying the Watts-Strogatz procedure with rewiring probability 1.



Random graph	
Diameter	4
Clustering coefficient	0.22
Correlation coefficient	0.42
Maximum transfer ($\times z$)	1
Minimum transfer ($\times z$)	1
Maximum degree	4
Minimum degree	4
Buffer size	0.047
Minimum ϵ	0.042
r_{min}	0.236
r_{max}	0.313

6.4 Overview of graph structures

The numerical examples are coherent with the theoretical results from the previous chapter. We have kept the parameters (R, ω_H, ω_L) fixed, and calculated two r-values, an upper bound, r_{min} that guarantees full contagion given (R, ω_H, ω_L) and an upper bound on r_{max} such that a crisis can spread to a non-empty subset of nodes. The r_{min} gives an ordering of the graphs we have considered, where larger r_{min} in some sense indicate a less robust market structure. The smaller the difference between $|r_{max} - r_{min}|$, the more likely is a crisis to reach the whole economy if it starts spreading. If $r_{max} = r_{min}$ spread to the whole economy is guaranteed if the spillover exceeds the buffer in any region.

An overview of the results sorted from less robust to more robust (r_{max} in descending order) is presented in table 6.1.

Graph type	r_{max}	r_{min}	ϵ	No. of edges
Max.corr.lin.graph	0.765	0.335	0.054	19
Random tree	0.570	0.260	0.047	19
Max.corr.lattice	0.522	0.043	0.007	40
Min.corr.lin.graph	0.335	0.313	0.058	19
Star	0.335	0.149	0.025	19
Random graph	0.313	0.236	0.042	40
Min.corr.lattice	0.176	0.176	0.030	40
Regular bipartite graph	0.102	0.102	0.102	40

Table 6.1: Overview of graph structures

We see that the three most important results from the previous chapter are supported by numerical calculations. The maximum correlation linear graph has the highest r_{min} and r_{max} of the graphs tested. It is thus the least robust by both measures. Furthermore we see that r_{min} is lower for the star than for other trees. This comes from the shock absorbing properties of the center node. For the regular bipartite graph, r_{max} is lower than for the rest of the graphs. This reflects the property that the worst case in other graphs (with the same number of edges) is worse than for this graph. The only somewhat surprising result is that r_{min} is lower for the maximum correlation lattice than for any other graph. The reason for this is that the solution of the minimum flow distribution problem is not unique, and the solution found by algorithm 1 is one where the transfers are unequally distributed between edges, and in that sense suboptimal. If we improved the algorithm such that the solution chosen was one where flow was distributed as equally as possible between edges, we would get a different and more realistic result. Since this thesis has focused on theoretical analysis, and not on the algorithmic aspects of the problem, this has not been given priority.

Chapter 7

Concluding remarks

In this thesis the model presented by Allen and Gale has been analyzed for different network structures. In [2], Allen and Gale conclude that complete markets are more robust than incomplete markets. Furthermore they suggest that the network structure of incomplete markets affects the robustness properties of the market. Exploring the robustness properties of several incomplete networks, this thesis confirms that the network structure plays a crucial role in determining a markets ability to absorb regional liquidity shocks.

7.1 Summary of main results

It has been argued that the robustness analysis of market structures can be reduced to two mathematical problems: a flow minimization problem and a breadth-first search algorithm. By considering properties of the mathematical problems, we have been able to show properties of certain market structures. The two most important results we have shown are

- The maximum correlation linear graph is the least robust market structure
- The k-regular bipartite graph is the most robust market structure with $nk/2$ edges

The first result was derived by considering the minimum flow distribution problem (4.6) for general graphs and deriving an upper bound on the total flow $\sum_{a \in A} f(a)$ and the single edge value $\min_{a \in A} f(a)$. Then we showed that the minimum flow distribution problem associated to the maximum correlation linear graph attains those upper bounds. This showed that the interdependencies in terms of interbank deposits must be larger for the maximum correlation linear graph than for any other market structure. We then considered the spillover effect represented by the cost function $\mathcal{C}(a)$. We were able to find an upper bound on its minimum value, and we could show that the spillover effects associated to the maximum linear correlation graph are at least as large as the upper bound on the minimum value. We could therefore conclude that if the cost-capacity condition is violated for all arcs in any network, it is violated for all arcs in the network associated to the maximum linear correlation graph.

The second result was derived with a similar approach. We derived lower bounds for the minimum flow distribution problem associated to graphs with $nk/2$ edges, and showed that the lower bounds were attained for the k-regular bipartite graph. Then, by considering the cost-capacity condition for the problem associated to a graph G, we showed that we can find a set of parameters γ such that the condition is violated for a non-empty subset of arcs in G if and only if G is not the k-regular bipartite graph.

7.2 Further research

This thesis takes a small step towards a better understanding of how the market structure affects the robustness of the market within the framework provided by Allen and Gale. Although we restricted our focus to robustness in the Allen and Gale model for financial contagion, there are many interesting aspects that have not been discussed and investigated in this thesis.

7.2.1 Robustness

The notion of robustness is an important discussion that has not been raised in this thesis. An illustration is the robustness result we have derived for the k-regular bipartite graph. We argue that it is the most robust graph structure, and it certainly is by many measures, among others on average. Yet we might find other graphs where certain nodes have a greater shock-absorbing quality than nodes in the k-regular bipartite graph. The difference between robustness towards random and targeted attacks have briefly been mentioned in the thesis, but is one of the problems that one would need to address for more general robustness analysis. Another problem in defining robustness is that for symmetric graphs, all regions will be bankrupt if the spillover exceeds the buffer for at least one region, since the regions have identical positions in the graph. For asymmetric graphs, the consequence of a liquidity shock depends on the starting region and a crisis does not necessarily spread to the whole economy. It is not obvious how these different effects can be captured in a definition of robustness, and this discussion would be important for further work.

7.2.2 Generalization of results

We have found an optimality result for graphs with $nk/2$ edges. A natural next step would be to describe optimality conditions for any fixed number of edges. There seem to be two key features that contribute strongly to the robustness of a graph in the model

- Regularity (degree distribution)
- Low correlation between node types

Regularity and low correlation are conditions that together ensure that the insurance that regions hold against uncertainty of future demand can as spread out as possible, which in turn results in smaller interdependencies between regions. Although this might be intuitively clear, and supported by the results in this thesis, formalizing the intuition does not seem to be straightforward, and this could be an interesting area of future work.

7.2.3 Real-world networks

In this thesis we consider mainly optimal and worst-case networks, none of which we can expect to observe in the real world. Apart from the theoretical interest in the networks analyzed, they may serve as benchmarks providing upper and lower bounds on robustness. To better understand complex real-world networks, it would be interesting to describe robustness in terms of network properties such as diameter, clustering, cohesiveness and correlation. The program code written for this thesis, can be a starting point for simulations to analyze these networks. By understanding robustness properties of financial networks that we observe, policy makers, financial intermediaries and investors might be able to redesign the networks such that, in the future, financial crises are less likely to occur and have smaller impact on the global economy. Complex real-world networks are therefore one of the most important directions to look into for further research in this field.

Bibliography

[1] F. Allen and A. Babus. Networks in finance. In P. Kleindorfer, Y. Wind, and R. Gunther, editors, *Capital Flows and the International Financial System*. Wharton School Publishing, 2009.

[2] F. Allen and D. Gale. Financial contagion. *The Journal of Political Economy*, 108(1):1–33, 2000.

[3] J. Bondy and U. Murty. *Graph Theory*. Springer, 2008.

[4] M. Bordo and B. Eichengreen. Is our current international economic environment unusually crisis prone? In D. Gruen and L. Gower, editors, *Capital Flows and the International Financial System*, RBA Annual Conference Volume. Reserve Bank of Australia, 1999.

[5] M. C. G. Bordo and E. White. *The Defining Moment: The Great Depression and the American Economy in the Twentieth Century*. Chicago: University Chicago Press, 1998.

[6] G. J. Caprio and D. Klingebiel. Bank insolvencies : cross-country experience. Policy Research Working Paper Series 1620, The World Bank, July 1996.

[7] A. H. Dekker and B. D. Colbert. Network robustness and graph topology. In *Proceedings of the 27th Australasian conference on Computer science - Volume 26*, ACSC '04, pages 359–368, Darlinghurst, Australia, Australia, 2004. Australian Computer Society, Inc.

[8] D. Diamond. Banks and liquidity creation: A simple exposition of the diamond-dybvig model. *Economic Quarterly*, 93(2):189–200, 2007.

[9] D. W. Diamond. Financial intermediation and delegated monitoring. *Review of Economic Studies*, 51(3):393–414, July 1984.

[10] D. W. Diamond and P. H. Dybvig. Bank runs, deposit insurance, and liquidity. *The Journal of Political Economy*, 91(3):pp. 401–419, 1983.

[11] Eurostat. European economic statistics. Technical report, European Commission, Januray 2011.

[12] I. M. Fund. Global financial stability report. Technical report, International Monetary Fund, October 2010.

[13] P. Gai and S. Kapadia. Contagion in financial networks. Technical Report Working paper 383, Bank of England, 2010.

[14] G. Gorton and A. Winton. Financial intermediation. In G. Constantinides, M. Harris, and R. M. Stulz, editors, *Handbook of the Economics of Finance*, volume 1 of *Handbook of the Economics of Finance*, chapter 8, pages 431–552. Elsevier, 2003.

[15] M. O. Jackson. *Social and Economic Networks*. Princeton University Press, Aug. 2008.

[16] C. Kindleberger. *The World in Depression*. Berkeley: University of California Press, 1973.

[17] J.-J. Laffont and D. Martimort. *The Theory of Incentives : The Principal-Agent Model*. Princeton University Press, Dec. 2001.

[18] N. B. I. Management. Third quarter 2010. Technical report, Norwegian Bank Investment Management, 2010.

[19] J. Nash. Non-Cooperative Games. *The Annals of Mathematics*, 54(2):286–295, Sept. 1951.

[20] N. Nisan, T. Roughgarden, E. Tardos, and V. V. Vazirani. *Algorithmic Game Theory*. Cambridge University Press, New York, NY, USA, 2007.

[21] G. Paul, T. Tanizawa, S. Havlin, and H. E. Stanley. Optimization of robustness of complex networks. *European Physical Journal B*, 38:187–191, Mar. 2004.

[22] L. Saez and X. Shi. Liquidity pools, risk sharing, and financial contagion. *Journal of Financial Services Research*, 25:5–23, 2004. 10.1023/B:FINA.0000008662.59653.33.

[23] J. Schildbach. Direct fiscal cost of the financial crisis. Technical report, Deutsche Bank Research, May 2010.

[24] O. Sprague. *History of Crises under the National Banking System*. Washington: Government Printing Office, 1910.

[25] F. Vega-Redondo. *Complex social networks*. Cambridge University Press, 2007.

[26] J. Watson. *Strategy: An Introduction to Game Theory*. New York: Norton, 2002.

[27] D. J. Watts and S. H. Strogatz. Collective dynamics of 'small-world' networks. *Nature*, 393(6684):440–442, June 1998.

Appendix A

Flow tables

Each flow table gives a vector w that solves the circulation problem. Since $w(a) = w(-a)$, we report the flow for the underlying edge, which includes both a and $-a$. Redundant edges are included in these tables. Flow added to redundant edges in simulations has not been included

Flow in max.corr.lin.graph	
Edge	Flow
$(1, 2)$	0.1
$(2, 3)$	0.2
$(3, 4)$	0.3
$(4, 5)$	0.4
$(5, 6)$	0.5
$(6, 7)$	0.6
$(7, 8)$	0.7
$(8, 9)$	0.8
$(9, 10)$	0.9
$(10, 11)$	1
$(11, 12)$	0.9
$(12, 13)$	0.8
$(13, 14)$	0.7
$(14, 15)$	0.6
$(15, 16)$	0.5
$(16, 17)$	0.4
$(17, 18)$	0.3
$(18, 19)$	0.2
$(19, 20)$	0.1

Flow in min.corr.lin.graph	
Edge	Flow
$(1, 2)$	0.1
$(2, 3)$	0
$(3, 4)$	0.1
$(4, 5)$	0
$(5, 6)$	0.1
$(6, 7)$	0
$(7, 8)$	0.1
$(8, 9)$	0
$(9, 10)$	0.1
$(10, 11)$	0
$(11, 12)$	0.1
$(12, 13)$	0
$(13, 14)$	0.1
$(14, 15)$	0
$(15, 16)$	0.1
$(16, 17)$	0
$(17, 18)$	0.1
$(18, 19)$	0
$(19, 20)$	0.1

Flow in star graph	
Edge	Flow
$(1,2)$	0.1
$(2,3)$	0.1
$(3,4)$	0.1
$(4,5)$	0.1
$(5,6)$	0.1
$(6,7)$	0.1
$(7,8)$	0.1
$(8,9)$	0.1
$(9,10)$	0.1
$(10,11)$	0.1
$(11,12)$	0.1
$(12,13)$	0.1
$(13,14)$	0.1
$(14,15)$	0.1
$(15,16)$	0.1
$(16,17)$	0.1
$(17,18)$	0.1
$(18,19)$	0.1
$(19,20)$	0.1

Flow in random tree	
Edge	Flow
$(1,2)$	0
$(2,3)$	0.1
$(3,4)$	0.3
$(4,5)$	0.2
$(5,6)$	0.1
$(6,7)$	0.1
$(7,8)$	0
$(8,9)$	0.1
$(9,10)$	0
$(10,11)$	0.1
$(11,12)$	0
$(12,13)$	0.1
$(6,14)$	0.1
$(5,15)$	0.2
$(15,16)$	0.1
$(16,17)$	0
$(17,18)$	0.1
$(1,19)$	0.1
$(3,20)$	0.1

Flow in reg.bipart. graph		**Flow in min.corr.lattice**	
Edge	Flow	Edge	Flow
$(1,2)$	0.025	$(1,2)$	0.05
$(2,3)$	0.025	$(1,3)$	0
$(1,4)$	0.025	$(2,3)$	0.05
$(3,4)$	0.025	$(2,4)$	0
$(2,5)$	0.025	$(3,4)$	0.05
$(4,5)$	0.025	$(3,5)$	0
$(3,6)$	0.025	$(4,5)$	0.05
$(5,6)$	0.025	$(4,6)$	0
$(4,7)$	0.025	$(5,6)$	0.05
$(6,7)$	0.025	$(5,7)$	0
$(5,8)$	0.025	$(6,7)$	0.05
$(7,8)$	0.025	$(6,8)$	0
$(6,9)$	0.025	$(7,8)$	0.05
$(8,9)$	0.025	$(7,9)$	0
$(7,10)$	0.025	$(8,9)$	0.05
$(9,10)$	0.025	$(8,10)$	0
$(8,11)$	0.025	$(9,10)$	0.05
$(10,11)$	0.025	$(9,11)$	0
$(9,12)$	0.025	$(10,11)$	0.05
$(11,12)$	0.025	$(10,12)$	0
$(10,13)$	0.025	$(11,12)$	0.05
$(12,13)$	0.025	$(11,13)$	0
$(11,14)$	0.025	$(12,13)$	0.05
$(13,14)$	0.025	$(12,14)$	0
$(12,15)$	0.025	$(13,14)$	0.05
$(14,15)$	0.025	$(13,15)$	0
$(13,16)$	0.025	$(14,15)$	0.05
$(15,16)$	0.025	$(14,16)$	0
$(14,17)$	0.025	$(15,16)$	0.05
$(16,17)$	0.025	$(15,17)$	0
$(1,18)$	0.025	$(16,17)$	0.05
$(15,18)$	0.025	$(16,18)$	0
$(17,18)$	0.025	$(17,18)$	0.05
$(2,19)$	0.025	$(1,19)$	0
$(16,19)$	0.025	$(17,19)$	0
$(18,19)$	0.025	$(18,19)$	0.05
$(1,20)$	0.025	$(1,20)$	0.05
$(3,20)$	0.025	$(2,20)$	0
$(17,20)$	0.025	$(18,20)$	0
$(19,20)$	0.025	$(19,20)$	0.05

Flow in max.corr.lattice	
Edge	Flow
$(1,2)$	0
$(1,3)$	0.16
$(2,3)$	0.018
$(2,4)$	0.12
$(3,4)$	0
$(3,5)$	0.079
$(4,5)$	0.011
$(4,6)$	0.011
$(5,6)$	0
$(5,7)$	0.011
$(6,7)$	0.011
$(6,8)$	0.079
$(7,8)$	0
$(7,9)$	0.12
$(8,9)$	0.018
$(8,10)$	0.16
$(9,10)$	0
$(9,11)$	0.24
$(10,11)$	0.021
$(10,12)$	0.24
$(11,12)$	0
$(11,13)$	0.16
$(12,13)$	0.018
$(12,14)$	0.12
$(13,14)$	0
$(13,15)$	0.079
$(14,15)$	0.011
$(14,16)$	0.011
$(15,16)$	0
$(15,17)$	0.011
$(16,17)$	0.011
$(16,18)$	0.079
$(17,18)$	0
$(1,19)$	0.24
$(17,19)$	0.12
$(18,19)$	0.018
$(1,20)$	0.021
$(2,20)$	0.24
$(18,20)$	0.16
$(19,20)$	0

Flow in random graph	
Edge	Flow
$(3,7)$	0
$(5,9)$	0
$(6,9)$	0.1
$(1,10)$	0.1
$(2,10)$	0
$(4,10)$	0
$(2,11)$	0.1
$(4,12)$	0
$(5,14)$	0.1
$(6,14)$	0
$(1,15)$	0
$(4,15)$	0.1
$(9,15)$	0
$(10,15)$	0
$(5,16)$	0
$(11,16)$	0
$(13,16)$	0.1
$(3,17)$	0
$(6,17)$	0
$(11,17)$	0
$(12,17)$	0.1
$(14,17)$	0
$(3,18)$	0
$(4,18)$	0
$(7,18)$	0.1
$(8,18)$	0
$(12,18)$	0
$(13,18)$	0
$(4,19)$	0
$(6,19)$	0
$(8,19)$	0.1
$(10,19)$	0
$(15,19)$	0
$(3,20)$	0.1
$(5,20)$	0
$(6,20)$	0
$(10,20)$	0
$(12,20)$	0
$(13,20)$	0
$(15,20)$	0

Appendix B

MATLAB Code

Except the Watts-Strogatz rewiring procedure and the conversion from adjacency matrix to incidence matrix, all code has been written specifically for this thesis. For graph drawing, I have used the package graphViz4Matlab, but written a new method and also slightly modified other code segments to make it to streamline it for our problem. The graphViz4Matlab-code is not included in this appendix. The implementation of the Watts and Strogatz rewiring procedure is taken from CONTEST toolbox by University of Strathclyde[1]. Conversion from adjacency matrix to incidence matrix of a graph uses code written by Ondrej Sluciak[2].

[1] http://www.mathstat.strath.ac.uk
[2] http://www.mathworks.com/matlabcentral

Program code B.1: Find properties of a given graph

```
function [A v rlist] = testRun(graphtype, correlation, varargin)

oh = 0.35;
ol = 0.15;
gamma = (oh+ol)/2;
z = oh-gamma;
R=2;
[x,y,c1]=firstBest(R,oh,ol);

[A v clust betw dist correl]= genGraph(graphtype, correlation, varargin{1},
    varargin{2}, varargin{3});

A = A>0;
A = solveLP(A,v);
A=A*z;

rlist = rlim(A,x,y,c1,gamma,R);
minr=min(rlist);
b=buffer(minr,x,gamma,c1,R);
mineps = (minr*x-minr*c1/R*(1-gamma))/(c1-c1*minr/R);
maxr=max(rlist);
drawGraph(A,v);
cl = mean(clust);
diameter = max(max(dist));
A = full(A);
maxA = max(max(A));
minA = min(A(A>0.0001));
A = sparse(A);

fprintf(1,'Clustering coefficient is:  %d \n',cl);
fprintf(1,'Diameter is:  %d \n',diameter);
fprintf(1,'Correlation coefficient is:  %d \n',correl);
fprintf(1,'Minimum epsilon to exceed the buffer is:  %d \n',mineps);
fprintf(1,'Maximum r to cause contagion:  %d \n',minr);
fprintf(1,'Maximum r to cause spillover to exceed buffer in any node:  %d \n
    ',maxr);
fprintf(1,'Buffer size is:  %d \n',b);
fprintf(1,'Max transfer is:  %d \n',maxA);
fprintf(1,'Min transfer is:  %d \n',minA);
```

Program code B.2: Optimal contract

```
function [x,y,c1,c2] = firstBest(R,oh,ol)
%Solves the first-best optimal contract given input R omegaH and omegaL
%The solution is found directly from setting the derivative of the problem
%to 0.
gamma = (oh+ol)/2;
konst = (1-gamma)/(gamma*R);
c1diff = -konst;

c2 = fzero(@(x) ud(x,gamma,R, c1diff, konst), 1);

x = (1-gamma)*c2/R;
y = 1-x;
c1 = y/gamma;

function uderiv = ud(x,gamma,R,c1diff, konst)
uderiv=c1diff*gamma/((1/gamma-konst*x)^2)+(1-gamma)/x^2;
```

Program code B.3: Find optimal deposits

```
function A = solveLP(A,v)
%Input: adjacency matrix A and a vector v. Solves the min.flow distribution
%problem
C = incidence(A)';
l = length(C);
f = ones(1, l);
B = [C;-C; -eye(l)];
k = v+((v==0)*(-1));
z = zeros(1,l);
k = [k -k z];

%solves the LP problem
w = linprog(f,B,k);
%Creates a weight matrix corresponding to the adjacency matrix
[i j] = find(A);
A = sparse(i,j,w);
A = (A+A');
%Add flow to redundant edges
%err = 0.00001
%for i=1:length(A)
%     mini = min(A(i,A(i,:)>err));
%     for j = 1:length(A)
%         if(A(i,j)>0 && A(i,j)<err)
  %            A(i,j)=mini;
%  A(i,j)=0
%         end
%     end
%end
```

Program code B.4: Find r-values

```
function rlist= rlim(W,x,y,c1,gamma,R)
%Function that calculates the r values that will cause bankruptcy in bank j
%given an initial bankruptcy in i. Returns a list of r values that
    corresponds to the
%edges in the graph.
m = sum(sum(W>0));
rlist = zeros(1,m);
n = length(W);
temp4 = ((1-gamma)*c1)/R;
count=1;
for i=1:n
    for j=1:n
        wij = W(i,j);
        if(wij>0)
        sumwij = sum(W(i,:));
        t1 = wij*c1;
        t2 = (y+sumwij*c1)/(1+sumwij);
        nom = t1-wij*t2;
        t3 = (x*wij)/(1+sumwij);
        denom = t3+x-temp4;
        rlist(count)=nom/denom;
        count = count +1;
        end
    end
end
```

Program code B.5: Generate graph

```
function [A v clust betw dist correl] = genGraph(type, corr, n, varargin)
%Function that generates different graph types. It takes graph type,
%correlation pattern, number of nodes, average degree and rewiring
%probability as parameters. It returns the adjacency matrix of the graph,
%the vector v with node types, and clustering coefficient, betweenness,
%distance and correlation coefficient of the graph.
if(strcmp(type,'line'))
    A = diag(ones(n-1,1),1);
    A = A+A';
elseif(strcmp(type,'star'))
    A = zeros(n);
    A(1,:)=ones(1,n);
    A = A'~=A;
elseif(strcmp(type,'new'))
    A = varargin(2);
    A=cell2mat(A);
elseif(strcmp(type,'bipartite'))
    k=varargin{1};
    A = bipartite(n,k);
elseif(strcmp(type,'lattice'))
    k = varargin{1};
    A = lattice(n,k);
elseif(strcmp(type,'rewire'))
    k = varargin{1};
    p = varargin{2};
    A = lattice(n,k);
    A = rewiring(A,p);
end

if(strcmp(corr,'max'))
    v = mod(1:n,2);
elseif(strcmp(corr,'min'))
    v = 1:n<=n/2;
elseif(strcmp(corr,'rand'))
    v = randperm(n)>n/2;
end

A = sparse(A);
A = +A;
clust = clustering_coefficients(A);
betw = betweenness_centrality(A);
dist = all_shortest_paths(A);
correl = correlation(A,v);
```

Program code B.6: Effect of bankruptcy

```
function list = bfsGraph(A,n, cond1, v,x,y,c1,r,R,z, gamma,draw)
if(draw)
[g colors]= drawGraph(A,v);
end
i=[];
list = zeros(1,n); %initialize list over bankrupt banks
spillover = zeros(1,n); %list of spillovereffects
if(cond1)
    b=buffer(r,x,gamma,c1,R);
    fprintf(1,'Buffersize is %f \n',b);
    list(1)=1;
    i(1)=1;
    colors(1)={'r'};
    if(draw)
        pause();
        redrawDyn(g,colors);
    end
    while(~isempty(i))
        change = 0;
        d = sum(full(A(i(1),:))); %degree of bankrupt node
        qi = findq(x,y,c1,z,r,d); %spillover from bankrupt node
        l=z*(c1-qi);
        for j=1:n
            if(A(i(1),j)==1 && list(j)==0)
                spillover(1,j) = spillover(1,j)+l;
                cond2 = spillover(1,j)>b;
                if(cond2)
                    change = 1;
                    list(j)=1;
                    colors(j)={'r'};
                    i(length(i)+1)=j;
                end
            end
        end
        i(1)=[];
        if(change && draw)
            pause();
            redrawDyn(g,colors);
        end
    end
end
end
```